Haim's Notes About

Introduction to Partial Differential Equations (2nd Ed)

by Gerald B. Folland

Important: Folland uses a convention on the Fourier transform in this book that falls into the minority category in terms of where one places the constant π. Here I follow his convention and so I strongly urge the reader to quickly glance at this convention in the section "Notations and Conventions" below before proceeding to any other part of this document.

Important: I reference many theorems from both this document and the book. If I merely write "Theorem X" then I'm referencing a theorem from this document. On the other hand, if I write "Theorem X from the book," then I'm referencing a theorem from the book.

1 Table of Contents

2 Notations and Conventions 2
3 Chapter 2 8
3.1 Invariance of the Wave Operator 8
3.2 Comment on Spherical Harmonics 8
Chapter 5 10
3.3 Wave Equation Cauchy Problem on Hyperplane 10
4 Chapter 6 11
4.1 Convergence in Sobolev Spaces Implies Convergence as Distributions 11
4.2 The H^0 Sobolev spaces 12
4.3 The Localized Sobolev Spaces 12
5 Chapter 8 13
5.1 The Zero Pseudodifferential Operator has *S* Negative Infinity Symbols (Proposition 8.11) 13
5.2 Equivalent Condition for Proper Support (Proposition 8.12) 16
5.3 A Useful Lemma for Amplitudes 18
5.4 Amplitude Kernels are Smooth Away from the Diagonal 20
5.5 Asymptotic Expansion of Amplitude Maps (Theorem 8.27) 22
5.6 Pseudodifferential Operators and Sobolev Spaces (Theorem 8.40) 27
5.7 Local Solvability of Properly Supported Elliptic Pseudodifferential Operators 29
5.8 Equivalent Definition of Wavefront Set (Theorem 8.56) 32

5.9 Change of Coordinates for Pseudodifferential Operators 35

2 Notations and Conventions

Definition 2.1: For functions $f \in L^1(\mathbb{R}^n)$, we define the **Fourier transform** of f as

$$\hat{f}(\xi) = \int e^{-2\pi i \xi \cdot x} f(x) dx.$$

The definition of the Fourier transform for tempered distributions follows from here.

Notation 2.2: For any tempered distribution u over $\mathbb{R}^n$, we denote its **inverse Fourier transform** by $\check{u}$. It is explicitly given by

$$\check{u}(x) = \hat{u}(-x).$$

Notation 2.3: We let D denote $(2\pi i)^{-1}$ times a derivative:

$$D = \frac{1}{2\pi i} \partial.$$

Notation 2.4: Suppose that Ω is an open subset and that $L = \sum_{|\alpha| \le k} a_\alpha \partial^\alpha$ is a linear partial differential operator over Ω with C^∞ coefficients. The **characteristic form** of L is the polynomial $\chi_L \in C^\infty(\Omega \times \mathbb{R}^n)$ given by

$$\chi_L(x, \xi) = \sum_{|\alpha| = k} a_\alpha(x) \xi^\alpha.$$

Notation 2.5: We will use $\langle \cdot \mid \cdot \rangle$ to denote inner product and $\langle \cdot\, , \ \cdot \rangle$ to denote distribution evaluations. In addition, in my distribution evaluations I will always put the distribution first and the test function second (i.e. if u is a distribution and ϕ is a test function, I always write $\langle u, \phi \rangle$ and never write $\langle \phi, u \rangle$).

Notation 2.6: For any point $h \in \mathbb{R}^n$ and any function or distribution u, we let $\tau_h u$ denote the translate of u in the direction h:

$$\tau_h u = u(x - h).$$

Notation 2.7: For any point $x \in \mathbb{R}^n$, we will denote its Euclidean length by $|x|$:

$$|x| = \sqrt{x_1^2 + \cdots + x_n^2}.$$

Notation 2.8: For any $x \in \mathbb{R}^n$ and any $r > 0$, we let $B_r(x)$ denote the open ball of radius r centered at x with respect to the Euclidean distance:

$$B_r(x) = \{y \in \mathbb{R}^n : |y - x| < r\}.$$

Notation 2.9: We let $\mathbb{Z}_+$ stand for the positive integers: $\mathbb{Z}_+ = \{1,2,3, \dots\}$.

Notation 2.10: For any $n \in \mathbb{Z}_+$, let $\mathcal{I}(n)$ denote the set of multi-indices of length n:

$$\mathcal{I}(n) = \{(\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n : \text{each } \alpha_k \geq 0\}.$$

Notation 2.11: In $\mathbb{R}^n$, for $j \in \{1, \dots, n\}$ we let e_j stand for the point/vector that has all zeros except a one in the j^{th} entry: $e_j = (0, \dots, 0, 1, 0, \dots, 0)$.

Notation 2.12: Suppose that $X \subseteq Y \subseteq \mathbb{R}^n$ are open sets. For any function $\phi \in C_c^\infty(X)$, we let $\phi^Y \in C_c^\infty(Y)$ denote the smooth extension of ϕ to Y obtained by setting $\phi \equiv 0$ on $Y \setminus X$. It's trivial to see then that $\operatorname{supp} \phi = \operatorname{supp} \phi^Y$ and hence ϕ^Y is indeed compactly supported as well.

We also extend this notation to other objects such as distributions in the obvious way.

Notation 2.13: Let $\alpha, \beta \in I(n)$ be multi-indices. Then

1.) $\alpha \leq \beta$ (resp. $\alpha < \beta$) means that each $\alpha_k \leq \beta_k$ (resp. $\alpha_k < \beta_k$).

2.) $\alpha!$ denotes $\alpha_1! \cdot \dots \cdot \alpha_n!$.

3.) $|\alpha|$ denotes $\alpha_1 + \cdots + \alpha_n$.

4.) For any $x \in \mathbb{R}^n$, x^α denotes $x_1^{\alpha_1} \cdot \dots \cdot x_n^{\alpha_n}$.

5.) For any sufficiently differentiable function or distribution f, $\partial^\alpha f$ denotes $\partial^{\alpha_1} \dots \partial^{\alpha_n} f$.

Notation 2.14: Let $\Omega \subseteq \mathbb{R}^n$ be an open set. We let the following denote the following spaces of *complex-valued* functions:

1.) $C^m(\Omega)$ denotes the space of k-times continuously differentiable functions over Ω. In particular, $C^\infty(\Omega)$ denotes the space of smooth functions.

2.) $C_c^m(\Omega)$ denotes the space of k-times continuously differentiable functions over Ω *with compact support*. Sometimes $C_c^\infty(\Omega)$ is also denoted by $\mathcal{D}(\Omega)$.

3.) We let $\mathcal{S}(\mathbb{R}^n)$ denotes the space of rapidly decreasing functions:

$$\mathcal{S}(\mathbb{R}^n) = \left\{\phi \in C^\infty(\mathbb{R}^n) : \left|x^\alpha \partial^\beta \phi(x)\right| < \infty \quad \forall \alpha, \beta \in \mathcal{I}(n)\right\}.$$

This space is called the **Schwartz space**.

Notation 2.15: Let $\Omega \subseteq \mathbb{R}^n$ be an open set. We let the following denote the following space of distributions:

1.) $\mathcal{D}'(\Omega)$ denotes the space of distributions over Ω.

2.) $\mathcal{E}'(\Omega)$ denotes the space of distributions over Ω with compact support.

3.) $\mathcal{S}'(\mathbb{R}^n)$ denotes the space of tempered distributions over $\mathbb{R}^n$.

Definition 2.16: Suppose that $u \in \mathcal{D}'(\Omega)$ is a distribution. We define the **complex conjugate** of u to be the distribution given by: for all $\phi \in C_c^\infty(\Omega)$,

$$\langle \overline{u}, \phi \rangle = \overline{\langle u, \overline{\phi} \rangle}.$$

This matches with the usual complex conjugate for ordinary functions. I include this definition here since I've never seen it stated in any textbook.

Definition 2.17: Suppose that X is a measure space and that $1 \leq p < \infty$. We define the L^p norm of a measurable function $f : X \to \mathbb{C}$ to be

$$\|f\|_{L^p} = \left[\int |f|^p\right]^{1/p}.$$

In the case $p = 2$, we also define the L^2 inner product to be

$$\langle f, g \rangle_{L^2} = \int f\bar{g}.$$

We let $L^p(X)$ denote the set of functions for which the L^p norm is finite.

Definition 2.18: For any $s \in \mathbb{R}$ we let $H_s(\mathbb{R}^n)$ denote the Sobolev space given by

$$H_s(\mathbb{R}^n) = \left\{u \in \mathcal{S}'(\mathbb{R}^n) : (1 + |\xi|^2)^{s/2}\hat{u}(\xi) \in L^2(\mathbb{R}^n)\right\}.$$

In this space we impose the inner product

$$\langle u, v \rangle_s = \int (1 + |\xi|^2)^s \hat{u}(\xi)\overline{\hat{v}(\xi)}d\xi.$$

It's a standard exercise to show that this turns $H_s(\mathbb{R}^n)$ into a Hilbert space.

Notation 2.19: For any $s \in \mathbb{R}$, we let Λ^s denote the operator

$$\Lambda^s = [I - (2\pi)^{-2}\Delta]^{s/2}$$

which is explicitly given by

$$\Lambda^s u = \left[(1 + |\xi|^2)^{s/2}\hat{u}(\xi)\right]^{\vee} \qquad \forall u \in \mathcal{S}'(\mathbb{R}^n).$$

This operator is nice because it allows us to neatly write the norm of any $u \in H_s(\mathbb{R}^n)$ as the L^2 norm of $(\Lambda^s u)^\frown$, and the standard isometry from $H_{-s}(\mathbb{R}^n)$ to $[H_s(\mathbb{R}^n)]^*$ (the dual of $H_s(\mathbb{R}^n)$) as Λ^{-2s}.

Definition 2.20: A **symbol of order $\boldsymbol{m}$** $\in \mathbb{R}$ over an open set $\Omega \subseteq \mathbb{R}^n$ is a smooth function $p \in C^\infty(\Omega \times \mathbb{R}^n)$ such that for any compact subset $K \subseteq \Omega$ and any multi-indices $\alpha, \beta \in \mathcal{I}(n)$ there exists a constant $C_{\alpha,\beta,K} > 0$ such that

$$\left|D_x^\beta D_\xi^\alpha p(x, \xi)\right| \leq C_{\alpha,\beta,K}(1 + |\xi|)^{m-|\alpha|} \qquad \forall (x, \xi) \in K \times \mathbb{R}^n.$$

We denote the vector space of all symbols of order m over Ω by $S^m(\Omega)$. In addition, we define

$$S^{-\infty}(\Omega)=\bigcap_{m\in\mathbb{R}}S^{m}(\Omega)\qquad\text{and}\qquad S^{\infty}(\Omega)=\bigcup_{m\in\mathbb{R}}S^{\infty}(\Omega).$$

Definition 2.21: A function $p:\Omega\times\mathbb{R}^n\to\mathbb{C}$ is said to be **homogeneous of degree m in ξ** if there exists a $c>0$ such that $p(x,t\xi)=t^m p(x,\xi)$ whenever $t\geq 1$ and $|\xi|\geq c$.

Definition 2.22: A **pseudodifferential operator** is a linear map $P:C_c^\infty(\Omega)\to C^\infty(\Omega)$ of the form

$$Pu(x)=\int e^{2\pi i x\cdot\xi}p(x,\xi)\hat{u}(\xi)d\xi$$

where $p\in S^m(\Omega)$ is a symbol of order $m\in\mathbb{R}\cup\{\pm\infty\}$. With respect to this notation, we say that "p is a symbol of P" or alternatively that "P has p as a symbol" (P can have more than one symbol). For any symbol $p\in S^m(\Omega)$ we let $p(x,D)$ denote the pseudodifferential operator that it generates as above. We denote the vector space of all pseudodifferential operators of order m by $\Psi^m(\Omega)$. Note that for any pseudodifferential operator P as above, it's in fact unnecessary to check that P actually maps $C_c^\infty(\Omega)$ into $C^\infty(\Omega)$ since, as one can check, that is automatic from the above equation. In addition, one can also check that P is automatically continuous.

Definition 2.23: Suppose that $\{m_j : j=0,1,2,\ldots\}$ is a strictly decreasing sequence of real numbers and that $p\in S^{m_0}(\Omega)$ and $\{p_j\in S^{m_j}(\Omega) : j=0,1,2,\ldots\}$ are symbols of the indicated order. Then we say that the formal series $\sum_{j=0}^\infty p_j$ is an **asymptotic expansion** of p if for any $k\in\mathbb{Z}_+$,

$$p-\sum_{j=0}^{k-1}p_j\in S^{m_k}(\Omega).$$

In this case we write $p\sim\sum_{j=0}^\infty p_j$.

Definition 2.24: An **amplitude of order m** $\in\mathbb{R}$ over an open set $\Omega\subseteq\mathbb{R}^n$ is a smooth function $a\in C^\infty(\Omega\times\mathbb{R}^n\times\Omega)$ such that for any compact subset $K\subseteq\Omega$ and any multi-indices $\alpha,\beta,\gamma\in\mathcal{I}(n)$ there exists a constant $C_{\alpha,\beta,\gamma,K}>0$ such that

$$\left|D_x^\beta D_y^\gamma D_\xi^\alpha a(x,\xi,y)\right|\leq C_{\alpha,\beta,\gamma,K}(1+|\xi|)^{m-|\alpha|}\qquad\forall(x,\xi,y)\in K\times\mathbb{R}^n\times K.$$

We denote the vector space of all amplitudes of order m over Ω by $A^m(\Omega)$.

Notation 2.25: For any amplitude $a\in A^m(\Omega)$, we let $P_a:C_c^\infty(\Omega)\to C^\infty(\Omega)$ denote the operator given by

$$P_a u(x)=\iiint e^{2\pi i(x-y)\cdot\xi}a(x,\xi,y)u(y)dyd\xi.$$

Note that the fact that the above equation maps $C_c^\infty(\Omega)$ to $C^\infty(\Omega)$ is automatic; one can prove this by carefully justifying interchanging D_x partials under the integral signs using the inequalities in the definition of $A^m(\Omega)$. Furthermore, one can show that this map is continuous and hence has a distribution kernel.

Definition 2.26: Suppose that $\Omega \subseteq \mathbb{R}^n$ is an open set. Let $\pi_x : \Omega \times \Omega \to \Omega$ and $\pi_y : \Omega \times \Omega \to \Omega$ denote the projection maps $(x, y) \mapsto x$ and $(x, y) \mapsto y$ respectively. A subset $W \subseteq \Omega \times \Omega$ is said to be **proper** if the restrictions $\pi_x|_W : W \to \Omega$ and $\pi_y\big|_W : W \to \Omega$ are proper maps. Equivalently, for any compact subset $A \subseteq \Omega$, both

$$(A \times \Omega) \cap W \qquad \text{and} \qquad (\Omega \times A) \cap W$$

are compact subsets of $\Omega \times \Omega$.

Definition 2.27: Suppose that $\Omega \subseteq \mathbb{R}^n$ is an open set. A linear map $T : C_c^\infty(\Omega) \to C^\infty(\Omega)$ with distribution kernel K is said to be **properly supported** if $\operatorname{supp} K$ is a proper subset of $\Omega \times \Omega$.

Definition 2.28: Suppose that $S, T : C_c^\infty(\Omega) \to C^\infty(\Omega)$ are linear maps. Then we say that S is the **transpose** of T and write $S = T'$ if

$$\langle Tu, v\rangle = \langle u, T'v\rangle \qquad \forall u, v \in C_c^\infty(\Omega).$$

We say that S is the **adjoint** of T and write $S = T^*$ if

$$\langle Tu, \overline{v}\rangle = \langle u, \overline{T^* v}\rangle \qquad \forall u, v \in C_c^\infty(\Omega).$$

Definition 2.29: A symbol $p \in S^m(\Omega)$ of order m is said to be **elliptic of order $\boldsymbol{m}$** if for any compact subset $A \subseteq \Omega$ there exist constants c_A, C_A such that

$$|p(x, \xi)| \geq c_A |\xi|^m \qquad \forall x \in A \ \text{ and } \ \forall \xi \in \mathbb{R}^n : |\xi| \geq C_A.$$

A pseudodifferential operator $P \in \Psi^m(\Omega)$ of order m is said to be elliptic of order m if one of its symbols is elliptic of order m (since any two such symbols differ by a $S^{-\infty}(\Omega)$ symbol [c.f. Theorem 5.1 below], it's not hard to see that this is well defined.

Sometimes I will drop writing "of order m" and just say elliptic when the order is clear or irrelevant. Also, observe that elliptic linear differential operators with C^∞ coefficients are also elliptic in this sense as well.

Definition 2.30: Suppose that P is a pseudodifferential operator. Let $\tilde{P}$ be a properly supported pseudodifferential operator such that $(P - \tilde{P}) \in \Psi^{-\infty}(\Omega)$ (exists by Corollary 8.32 in the book). Then, a **parametrix** for P is a properly supported pseudodifferential operator Q (possibly of a different order) such that both $(\tilde{P}Q - I) \in \Psi^{-\infty}(\Omega)$ and $(Q\tilde{P} - I) \in \Psi^{-\infty}(\Omega)$ where I stands for the identity. It's not hard to see that this definition is independent of the $\tilde{P}$ that we choose (see Theorem 8.37 in the book).

It's a (nontrivial) fact that every elliptic pseudodifferential operator has a parametrix.

Definition 2.31: Suppose that $\Omega \subseteq \mathbb{R}^n$ is an open subset. We let the following denote:

$$T^0\Omega = \Omega \times (\mathbb{R}^n \setminus \{0\})$$

We say that a subset $V \subseteq T^0\Omega$ is **conic** if for any $(x,\xi) \in V$, we have that $(x, c\xi) \in V$ for all nonzero $c \in \mathbb{R}$.

Definition 2.32: Suppose that $p \in S^m(\Omega)$ is a symbol. We say that p is **elliptic of order $\boldsymbol{m}$** at $(x_0, \xi_0) \in T^0(\Omega)$ if there exists a conic neighborhood V of (x_0, ξ_0) and constants $c, C > 0$ such that

$$|p(x,\xi)| \geq c|\xi|^m \qquad \forall (x,\xi) \in V : |\xi| \geq C.$$

We say that a pseudodifferential operator $P \in \Psi^m(\Omega)$ is elliptic of order m at $(x_0, \xi_0) \in T^0(\Omega)$ if all of its symbols are elliptic of order m at $(x_0, \xi_0) \in T^0(\Omega)$ (by Proposition 8.11(b) in the book or Theorem 5.1.1 below, it suffices only to check one symbol).

Definition 2.33: Suppose that $P \in \Psi^m(\Omega)$ is a pseudodifferential operator. We define the **characteristic variety of order $\boldsymbol{m}$** of P to be the set

$$\operatorname{char}_m P = \{(x,\xi) \in T_0\Omega : P \text{ is not elliptic of order } m \text{ at } (x,\xi)\}.$$

It's easy to see that this is a closed cone in $T^0\Omega$.

Note: Folland doesn't include m in his notation or definition of the characteristic variety. I imagine that it's implicit.

Definition 2.34: Suppose that $p \in S^m(\Omega)$ is a symbol. We say that p is **smoothing at $(\boldsymbol{x_0}, \boldsymbol{\xi_0}) \in \boldsymbol{T^0\Omega}$** if there exists a conic neighborhood $V \subseteq T^0\Omega$ of (x_0, ξ_0) on which p is rapidly decreasing: for any $M > 0$ and any $\alpha, \beta \in I(n)$ there exists a constant $C_{M,\alpha,\beta} > 0$ such that

$$\left|D_x^\beta D_\xi^\alpha p(x,\xi)\right| \leq C_{M,\alpha,\beta}(1 + |\xi|)^{-M} \qquad \forall (x,\xi) \in V.$$

We say that a pseudodifferential operator $P \in \Psi^m(\Omega)$ is smoothing at $(x_0, \xi_0) \in T^0(\Omega)$ if all of its symbols are smoothing at $(x_0, \xi_0) \in T^0(\Omega)$ (by Proposition 8.11(b) in the book or Theorem 5.1.1 below, it suffices only to check one symbol).

Definition 2.35: Suppose that $P \in \Psi^m(\Omega)$ is a pseudodifferential operator. We define the **microsupport** of P to be the set

$$\mu\text{supp}(P) = \{(x,\xi) \in T^0\Omega : P \text{ is not smoothing at } (x,\xi)\}$$

It's easy to see that this is a closed cone in $T^0\Omega$.

Definition 2.36: Suppose that $u \in \mathcal{D}'(\Omega)$ is a distribution. We define the **wavefront set** of u as

$$WF(u) = \bigcap \{\operatorname{char}_0 P : P \in \Psi^0(\Omega) \text{ is properly supported and } Pu \in C^\infty(\Omega)\}.$$

This may be a different definition than what some people are used to, such as the one given in the book *Introduction to The Theory of Distributions (2nd Ed)* by Friedlander and Joshi, but Folland proves in his book that the two definitions are equivalent. See Section 5.8 below.

3 Chapter 2

3.1 Invariance of the Wave Operator

Here I solve Problem 2 in Section 2.A in the book, which asks to show that in $\mathbb{R}^2 = \{(x,t) : x, t \in \mathbb{R}\}$,

$$(\partial_t^2 - \partial_x^2)(u \circ T_\theta) = [(\partial_t - \Delta)u] \circ T_\theta. \tag{3.1}$$

where

$$T_\theta = \begin{bmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{bmatrix}.$$

We'll show this for $u \in \mathcal{S}(\mathbb{R}^2)$, from which the fact will follow for $u \in \mathcal{D}'(\mathbb{R}^2)$ by density.

Instead of doing it by brute force, let's solve this problem as conceptually as possible. Taking the Fourier transform of the left-hand side of (3.1) gives

$$(\partial_t^2 - \widehat{\partial_x^2)(}u \circ T_\theta) = (\tau^2 - \xi^2)(\widehat{u \circ T_\theta}) = (\tau^2 - \xi^2)\frac{1}{\det T_\theta}\hat{u} \circ \big([T_\theta^{-1}]^\top\big). \tag{3.2}$$

Using that

$$\cosh^2\theta - \sinh^2\theta = 1 \tag{3.3}$$

(this is why these are called the "hyperbolic" cosine and sine) it's quick to see that $\det T_\theta = 1$, $T_\theta^{-1} = T_{-\theta}$ (use Cramer's rule), and that $T_\theta^\top = T_\theta$ due to symmetry. Hence (3.2) is equal to $(\tau^2 - \xi^2)\hat{u} \circ T_{-\theta}$.

Now, a small computation using (3.3) shows that the columns of T_θ are orthonormal with respect to the Minkowski metric: $\langle(\tau,\xi),(\tau',\xi')\rangle_{\mathbb{R}^{1,1}} = \tau\tau' - \xi\xi'$, and hence T_θ preserves this Minkowski metric. This is simply a complicated way of saying that $(\tau^2 - \xi^2) = (\tau^2 - \xi^2) \circ T_\theta$. Hence (3.2) is further equal to

$$[(\tau^2 - \xi^2) \circ T_{-\theta}]\hat{u} \circ T_{-\theta} = [(\tau^2 - \xi^2)\hat{u}] \circ T_{-\theta} = \frac{1}{\det T_\theta}[(\tau^2 - \xi^2)\hat{u}] \circ \big([T_\theta^{-1}]^\top\big)$$
$$= ([(\partial_t^2 - \partial_x^2)u] \circ T_\theta)\hat{}.$$

Taking inverse Fourier transforms then gives (3.1) as desired.

3.2 Comment on Spherical Harmonics

Fix $n \geq 2$. Let B and S denote the unit open ball and unit sphere in $\mathbb{R}^n$ respectively centered at zero. In section 2.H, for every $k \geq 0$ the author defines the following spaces:

$$\mathcal{P}_k = \{\text{set of all homogeneous polynomials of degree } k \text{ over } \mathbb{R}^n\}$$

$$\mathcal{H}_k = \{P \in \mathcal{P}_k : \Delta P = 0\}$$

$$H_k = \{P|_S : P \in \mathcal{H}_k\}.$$

The author calls H_k the **spherical harmonics of degree $\boldsymbol{k}$**.

I want to point out that these H_k's are the eigenfunctions of the Laplacian over S. To see why, recall the standard formula for the Laplacian in $\mathbb{R}^n$ in spherical coordinates:

$$(3.4) \qquad \Delta = \frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r} + \Delta_{rS}$$

where Δ_{rS} is the Laplacian over the sphere rS. Now, take any $P|_S \in H_k$ where $P \in \mathcal{H}_k$. Since P is a homogeneous polynomial of degree k, in spherical coordinates it is of the form $P(rx) = Q(x)r^k$ for $r > 0$ and $x \in S$ where Q is some smooth function over S. Thus by the above equation we have that

$$0 = \Delta P = k(k-1)Q(x)r^{k-2} + (n-1)kQ(x)r^{k-2} + \Delta_{rS}[Q(x)r^k].$$

Now, restricting both sides of this equation to S where $r = 1$ gives us that

$$0 = k(k-1)Q(x) + k(n-1)Q(x) + \Delta_S[Q(x)].$$

Since $P|_S = Q$, we have that this implies that

$$\Delta_S P = -k(k+n-2)P.$$

So indeed, members of H_k are eigenfunctions of the spherical Laplacian Δ_S with eigenvalue $-k(k+n-2)$. From here we also quickly get that $H_k \perp H_j$ with respect to the inner product of $L^2(S)$ when $k \neq j$ by doing integration by parts twice over S. Furthermore, by Theorem 2.53 in the book we know that the H_k's span the whole Hilbert space $L^2(s)$ and so these must be all of the eigenfunctions of the spherical Laplacian (this requires a bit of Hilbert space theory to understand).

We also get a more intuitive proof of the following lemma that appears Lemma 2.62 in the book:

Lemma 3.5: *Suppose that $Y \in H_k$ and let $\tilde{Y} : \mathbb{R}^n \setminus \{0\} \to \mathbb{C}$ be the function*

$$\tilde{Y}(rx) = Y(x) \qquad \forall r > 0,\ \forall x \in S.$$

Then

$$\Delta\tilde{Y} = -k(k+n-2)r^{-2}\tilde{Y}.$$

Proof: For any $r > 0$ and any $x \in S$, we have by (3.4) above that (I leave out the details justifying the second equality below: it's a standard differential geometry fact/calculation):

$$\Delta\tilde{Y}(rx) = \Delta_{rS}\tilde{Y}(rx) = r^{-2}\Delta_S\tilde{Y}(x) = r^{-2}\Delta_S Y(x) = -r^{-2}k(k+n-2)Y(x)$$
$$= -r^{-2}k(k+n-2)\tilde{Y}(rx).$$

■

Chapter 5

3.3 Wave Equation Cauchy Problem on Hyperplane

Here I solve Problem 3 in Section 5.A in the book which asks us to do the following. Suppose that $\nu = (\nu', \nu_0) \in \mathbb{R}^n \times \mathbb{R}$ is a unit vector such that $|\nu'| < |\nu_0|$ and consider the hyperplane perpendicular to ν: $S = \{(x,t) \in \mathbb{R}^{n+1} : x \cdot \nu' + t\nu_0 = 0\}$.

First we show that that there is a linear map $T : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ that maps S to $\{(x,t) : t = 0\}$ of the form $T = T_2 T_1$ where (here "rotation" means "unitary matrix")

$$T_1(x,t) = (Rx, t), \quad \text{where } R \text{ is a rotation in } \mathbb{R}^n.$$

$$T_2(x,t) = (x_1', x_2, \dots, x_n, t'), \quad \text{where } \begin{bmatrix} x_1' \\ t' \end{bmatrix} = T_\theta \begin{bmatrix} x \\ t \end{bmatrix} \text{ where}$$

$$T_\theta = \begin{bmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{bmatrix}.$$

Let R be any unitary matrix that takes ν' to $(|\nu'|, 0, \dots, 0)$ and define T_1 as above. Observe that T_1 takes ν to $(|\nu'|, 0, \dots, 0, \nu_0)$. Since T_1 is unitary itself, it preserves inner products and hence take $S = \nu^\perp$ to $(T_1\nu)^\perp$. In other words,

$$T_1[S] = \{(x_1, \dots, x_n, t) : x_1|\nu'| + t\nu_0 = 0\}.$$

Hence we need to find a θ so that T_2 take this to $\{(x,t) : t = 0\}$. This will happen if and only if

$$\left\langle \begin{pmatrix} x_1 \\ t \end{pmatrix}, \begin{pmatrix} |\nu'| \\ \nu_0 \end{pmatrix} \right\rangle = 0 \quad \Longrightarrow \quad 0 = \left\langle T_\theta \begin{pmatrix} x_1 \\ t \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} x_1 \\ t \end{pmatrix}, T_\theta^\top \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} x_1 \\ t \end{pmatrix}, T_\theta \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle$$

This implication will hold if $(|\nu'|, \nu_0)$ is parallel to $T_\theta(0,1)$, or more precisely if there exists an $\alpha \in \mathbb{R}$ such that $\alpha(|\nu'|, \nu_0) = T_\theta(0,1)$. Writing this system out gives

$$\alpha|\nu'| = \sinh\theta, \tag{3.6}$$

$$\alpha\nu_0 = \cosh\theta. \tag{3.7}$$

Since $\cosh\theta > 0$, we need to choose α to have the same sign as ν_0. Since $\cosh^2\theta - \sinh^2\theta = 1$, we need to choose α so that $\alpha^2(\nu_0^2 - |\nu'|^2) = 1$. Since $|\nu'| < |\nu_0|$, this is possible by simply choosing

$$\alpha = \frac{\operatorname{sgn}\nu_0}{\sqrt{\nu_0^2 - |\nu'|^2}}.$$

With this α, the surjectivity of $\theta \mapsto \sinh\theta$ implies that there exists θ solving (3.6) and hence (3.7) because we choose the appropriate sign for α. This gives us the T_1, T_2, and hence $T = T_2 T_1$ that we wanted.

Next, suppose we want to solve the Cauchy problem

$$(\partial_t^2 - \Delta)u = 0, \qquad u|_S = f, \quad \partial_\nu u|_S = g.$$

Instead suppose that we can solve the Cauchy problem

$$(\partial_t^2 - \Delta)v = 0, \quad v|_{\{t=0\}} = f \circ T^{-1}, \quad \partial_t v|_{\{t=0\}} = \tilde{g},$$

where $\tilde{g}$ is to be determined. We set $u = v \circ T$. First let us show that u also solves the wave equation. We have that R preserves Δ (see Theorem 2.1 in the book) and hence T_1 preserves $(\partial_t^2 - \Delta)$ since it does not affect the t-axis. By Problem 2 in Section 2.H in the book (or see Section 3.1 above), T_2 preserves $\partial_t^2 - \partial_{x_1}^2$ and hence $(\partial_t^2 - \Delta)$ since it does not affect the x_i-axis for $2 \le i \le n$. Thus T preserves the $(\partial_t^2 - \Delta)$ and so indeed u satisfies the wave equation:

$$(\partial_t^2 - \Delta)u = (\partial_t^2 - \Delta)(v \circ T) = [(\partial_t^2 - \Delta)v] \circ T = 0.$$

Next let's show that u satisfies the proper boundary conditions. It's easy to see that $u|_S = f$ is indeed satisfied, so let us figure out what $\tilde{g}$ must be in order for $\partial_\nu u|_S = g$ to be satisfied. Observe that

$$\partial_\nu u = \partial_\nu (v \circ T) = dT(\nu) \cdot [(\nabla v) \circ T] = (\partial_{T\nu} v) \circ T.$$

Let us write $T\nu = (\omega', \omega_0)$, and note that $\omega_0 \neq 0$ since we know that the invertible T already takes the $(n-1)$-dimensional S to the $(n-1)$-dimensional $\{(x,t) : t = 0\}$ and hence cannot take $\nu \notin S$ to $\{(x,t) : t = 0\}$ as well. The above quantity can then be rewritten as

$$\left(\omega_0 \partial_{(0,\ldots,0,1)} v + \partial_{(\omega',0)} v\right) \circ T = \omega_0 (\partial_t v \circ T) + \left(\partial_{(\omega',0)} v\right) \circ T.$$

Now, we have that on $\{(x,t) : t = 0\}$, $\partial_{(\omega',0)} v = \partial_{(\omega',0)}(f \circ T^{-1})$. Thus if we set

$$\tilde{g} = \frac{1}{\omega_0}\left[g \circ T^{-1} - \partial_{(\omega',0)}(f \circ T^{-1})\right],$$

then indeed we'll get that $\partial_\nu u|_S = g$.

4 Chapter 6

4.1 Convergence in Sobolev Spaces Implies Convergence as Distributions

Here I prove the following quick and useful lemma:

Lemma 4.1: *Suppose that $s \in \mathbb{R}$. Suppose also that $\{u_j \in H_s(\mathbb{R}^n) : j \in \mathbb{Z}_+\}$ and $v \in H_s(\mathbb{R}^n)$ are such that $u_j \to v$ in $H_s(\mathbb{R}^n)$. Then $u_j \to v$ in $\mathcal{D}'(\mathbb{R}^n)$ as well.*

Proof: Take any test function $\phi \in C_c^\infty(\mathbb{R}^n)$ and observe that

$$\left|\langle u_j - v, \phi\rangle\right| = \left|\langle \widehat{u_j}(\xi) - \hat{v}(\xi), \hat{\phi}(-\xi)\rangle\right|$$

$$= \left|\int (1 + |\xi|^2)^{s/2}\left[\widehat{u_j}(\xi) - \hat{v}(\xi)\right](1 + |\xi|^2)^{-s/2}\hat{\phi}(-\xi) ds\right| \le \left\|u_j - v\right\|_s^2 \|\phi\|_{-s}^2.$$

The last quantity goes to zero as $j \to \infty$ by assumption, and so the lemma follows.

■

4.2 The H^0 Sobolev spaces

In this note we define the following space:

Definition 4.2: *Suppose $s \in \mathbb{R}$ and that $\Omega \subseteq \mathbb{R}^n$ is an open subset. We let $H_s^0(\Omega)$ denote the closure of $C_c^\infty(\Omega)$ in $H_s(\mathbb{R}^n)$. We impose on $H_s^0(\Omega)$ the inner product that it inherits from $H_s(\mathbb{R}^n)$ itself.*

Observe that since $H_s(\mathbb{R}^n)$ is a Hilbert space and $H_s^0(\Omega)$ is a closed subspace of it, $H_s^0(\Omega)$ is also a Hilbert space.

4.3 The Localized Sobolev Spaces

In this note we define and discuss the properties of the localized Sobolev spaces.

Definition 4.3: *Suppose that $s \in \mathbb{R}$ and that $\Omega \subseteq \mathbb{R}^n$ is an open subset. We let $H_s^{loc}(\Omega)$ denote the set of all distributions $u \in \mathcal{D}'(\Omega)$ such that for any bounded open subset Ω_0 of Ω satisfying $\overline{\Omega_0} \subseteq \Omega$, u agrees with an element of $H_s(\mathbb{R}^n)$ over Ω_0. We impose the topology described below on $H_s^{loc}(\Omega)$.*

By Proposition 6.13 in the book, we know that for any $u \in H_s^{\mathrm{loc}}(\Omega)$ and any $\phi \in C_c^\infty(\Omega)$, ϕu is in $H_s(\mathbb{R}^n)$ (technically we should be writing the "zero extension of ϕu to $\mathbb{R}^n$"). Thus we can impose the topology on $H_s^{\mathrm{loc}}(\Omega)$ generated by the family of seminorms

$$\{u \mapsto \|\phi u\|_s : \phi \in C_c^\infty(\Omega)\}. \tag{4.4}$$

This turns $H_s^{\mathrm{loc}}(\Omega)$ into a locally convex topological vector space. It's not hard to see that this space is Hausdorff. Furthermore, it is a Fréchet space as the following proposition shows.

Proposition 4.5: *Suppose that $s \in \mathbb{R}$ and that $\Omega \subseteq \mathbb{R}^n$ is an open subset. Then $H_s^{loc}(\Omega)$ is a Fréchet space.*

Proof: We will prove this proposition by proving the following

1. One only needs a countable subset of the seminorms described in (4.4) to generate the same topology on $H_s^{\mathrm{loc}}(\Omega)$, and hence this space is metrizable.
2. This space is complete.

First let's prove (1). Let $\mathcal{T}$ denote the topology on $H_s^{\mathrm{loc}}(\Omega)$ defined by (4.4). Let $\{W_k : k \in \mathbb{Z}_+\}$ be a precompact exhaustion of Ω (i.e. each W_k is an open subset of Ω such that $\overline{W_k} \subseteq W_{k+1}$ and $\Omega = \bigcup_{k=1}^\infty W_k$). For each $k \in \mathbb{Z}_+$, let $\phi_k \in C_c^\infty(\Omega)$ be such that $\phi \equiv 1$ on $\overline{W_k}$ and $\operatorname{supp} \phi \subseteq \mathrm{W}_{k+1}$. Let $\mathcal{T}'$ denote the topology on $H_s^{\mathrm{loc}}(\Omega)$ given by the following countable family of seminorms:

$$\{u \longmapsto \|\phi_k u\|_s : k \in \mathbb{Z}_+\}.$$

We will prove that $\mathcal{T} = \mathcal{T}'$. Consider the identity map $\mathrm{id} : (H_s^{\mathrm{loc}}(\Omega), \mathcal{T}') \to (H_s^{\mathrm{loc}}(\Omega), \mathcal{T})$. It's easy to see that this is continuous. To prove that the inverse identity map $\mathrm{id}^{-1} : (H_s^{\mathrm{loc}}(\Omega), \mathcal{T}) \to (H_s^{\mathrm{loc}}(\Omega), \mathcal{T}')$ is continuous, take any $\phi \in C_c^\infty(\Omega)$ and let $k \in \mathbb{Z}_+$ be such that $\operatorname{supp}\phi \subseteq W_k$. Observe that $\phi_k \equiv 1$ on $\operatorname{supp}\phi$ and so $\phi_k = \phi\phi_k$. Observe also that since $v \mapsto \phi v$ is a continuous $H_s(\mathbb{R}^n) \to H_s(\mathbb{R}^n)$ map (Proposition 6.12 in the book) there exists a constant $C_\phi > 0$ such that $\|\phi v\|_s \le C_\phi \|v\|_s$ for all $v \in H_s(\mathbb{R}^n)$. Thus for any $u \in H_s^{\mathrm{loc}}(\Omega)$ we have that

$$\|\phi u\|_s = \|\phi\phi_k u\|_s \le C_\phi \|\phi_k u\|_s.$$

So indeed id^{-1} is continuous. Hence the two topologies $\mathcal{T}$ and $\mathcal{T}'$ are the same and so we've proven (1).

Now let's prove (2). Suppose that $\{u_k : k \in \mathbb{Z}_+\}$ is a Cauchy sequence in $H_s^{\mathrm{loc}}(\Omega)$. For any open $W \subseteq \Omega$ such that $\overline{W} \subseteq \Omega$, let $\phi_W \in C_c^\infty(\Omega)$ be such that $\phi_W \equiv 1$ on $\overline{W}$. Observe that for any such W, $\{\phi_W u_k\}$ is Cauchy in $H_s(\Omega)$ and hence converges to some $v_W \in H_s(\mathbb{R}^n)$ as $k \to \infty$. Note that by Lemma 4.1 we also have that $\{\phi_W u_k\}$ converges to v_w in $\mathcal{D}'(\mathbb{R}^n)$ as well. Now, define the distribution $v \in H_s^{\mathrm{loc}}(\Omega)$ as follows: for any open set W as above let v over W be equal to the restriction of v_W to W. We will now show that this is well defined and that $u_k \to v$ in $H_s^{\mathrm{loc}}(\Omega)$. To prove this first claim, take any open sets W and W' as above such that $W \cap W' \neq \emptyset$. Then for any $\psi \in C_c^\infty(W \cap W')$ we have that

$$\langle v_W, \psi\rangle = \lim_{k\to\infty} \langle \phi_W u_k, \psi\rangle = \lim_{k\to\infty} \langle u_k, \phi_W \psi\rangle = \lim_{k\to\infty} \langle u_k, \psi\rangle.$$

Since a similar calculation shows that this is also equal to $\langle v_{W'}, \psi\rangle$, we have that v is indeed well defined. Finally, let's show that $u_k \to v$ in $H_s^{\mathrm{loc}}(\Omega)$. Take any $\phi \in C_c^\infty(\Omega)$. Let W be an open set as above such that $\operatorname{supp}\phi \subseteq W$ and observe that $\phi = \phi\phi_W$. Then, since multiplying by ϕ is a continuous operation in $H_s(\mathbb{R}^n)$ (c.f. previous paragraph) we have that $\phi_W u_k$ converging to v_W in $H_s(\mathbb{R}^n)$ implies that

$$\phi u_k = \phi\phi_W u_k \to \phi v_W = \phi v \quad \text{in} \quad H_s(\mathbb{R}^n) \quad \text{as} \quad k \to \infty.$$

From here we see that indeed $u_k \to v$ in $H_s^{\mathrm{loc}}(\Omega)$ and so (2) is proven.

∎

5 Chapter 8

5.1 The Zero Pseudodifferential Operator has *S* Negative Infinity Symbols (Proposition 8.11)

Note: The version of this proposition in the book has a mistake. I describe the correct version here and I'd like to thank the author of this book for pointing out how the mistake needs to be fixed.

In this section I work through the proof of the following theorem that appears in this book by putting it into my own words.

Theorem 5.1: *Suppose that $\Omega \subseteq \mathbb{R}^n$ is an open set and that $P \in \Psi^m(\Omega)$ is a pseudodifferential operator over Ω equal to zero as a map $P : C_c^\infty(\Omega) \to C^\infty(\Omega)$ (i.e. $Pu = 0$ for any $u \in C_c^\infty(\Omega)$). Let $p \in C^\infty(\Omega \times \mathbb{R}^n)$ be a symbol of P. Then*

1.) $p \in S^{-\infty}(\Omega)$.

2.) If Ω is dense in $\mathbb{R}^n$, then in fact $p \equiv 0$. If Ω is not dense in P, then p can be chosen so that $p \neq 0$.

Proof: First let's prove (1). Let $p \in C^\infty(\Omega \times \mathbb{R}^n)$ be a symbol of P. For every fixed $x \in \Omega$, let $p_2^\vee(x,z) \in \mathcal{D}'(\mathbb{R}^n)$ denote the distribution that is the inverse Fourier transform of the smooth function $z \mapsto p(x,z)$. It's explained in the book before this theorem that $p_2^\vee(x, x-z)|_{z\in\Omega}$ is related to $P = p(x,D)$ by the equation

$$p(x,D)u = \langle p_2^\vee(x,x-z)|_{z\in\Omega}, u(z)\rangle \qquad \forall u \in C_c^\infty(\Omega)$$

where the right-hand side of this equation is regarded as a function of $x \in \Omega$. Now, I claim that for any fixed $x \in \Omega$, the distribution $p_2^\vee(x,z) = 0$ on the open set $\{x\} - \Omega$. To see this, take any $\phi \in C_c^\infty(\{x\} - \Omega)$, let $\phi^{\mathbb{R}^n}$ denote its zero extension to $\mathbb{R}^n$ (see Notation 2.12), and then observe that

$$\langle p_2^\vee(x,z)|_{z\in\{x\}-\Omega}, \phi(z)\rangle = \langle p_2^\vee(x,z), \phi^{\mathbb{R}^n}(z)\rangle = \langle p_2^\vee(x,x-z), \phi^{\mathbb{R}^n}(x-z)\rangle$$

$$= \langle p_2^\vee(x,x-z)|_{z\in\Omega}, \phi^{\mathbb{R}^n}(x-z)\big|_{z\in\Omega}\rangle = p(x,D)\left[\phi^{\mathbb{R}^n}(x-z)\big|_{z\in\Omega}\right] = 0.$$

Another way to state this is that $p_2^\vee(x,z)$ is equal to zero on the set

$$\mathcal{O} = \{(x,z) \in \Omega \times \mathbb{R}^n : z \in \{x\} - \Omega\}$$

in the sense that for any fixed x, $p_2^\vee(x,z)$ is zero on the set of $z \in \mathbb{R}^n$ that satisfy $(x,z) \in \mathcal{O}$ (i.e. on the x-slice of $\mathcal{O}$). Notice that $\mathcal{O}$ is an open subset of $\Omega \times \mathbb{R}^n$ since if one lets $G : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ denote the continuous function $G(x,z) = x - z$, then $\mathcal{O} = (\Omega \times \mathbb{R}^n) \cap G^{-1}[\Omega]$.

Next, by Theorem 8.8 in the book we know that for any $\alpha \in \mathcal{I}(n)$ such that $|\alpha| > m + n$, for any fixed $x \in \Omega$ the distribution $z^\alpha p_2^\vee(x,z)$ is a continuous function. Let $f_\alpha : \Omega \times \mathbb{R}^n \to \mathbb{C}$ denote the function whose value at any point $(x_0, z_0) \in \Omega \times \mathbb{R}^n$ is equal to the value of $z^\alpha p_2^\vee(x_0, z)$ at $z = z_0$. By Theorem 8.8 we furthermore know that f_α is class $C^j(\Omega \times \mathbb{R}^n)$ for any $j \in \mathbb{Z}_+ \cup \{0\}$ such that $|\alpha| > m + n + j$ and that for any such j the partials of f_α of order $\leq j$ are bounded on sets of the form $K \times \mathbb{R}^n$ where K is a compact subset of Ω. By the previous paragraph we also know that f_α is zero on $\mathcal{O}$, which is an open neighborhood of $\Omega \times \{0\}$ in $\Omega \times \mathbb{R}^n$. Thus, by dividing the above f_α's be z^α, we see that for any fixed $x \in \Omega$ the distribution $p_2^\vee(x,z)$ is in fact in $C^\infty(\mathbb{R}^n)$. As with the f_α's, let's let $p_2^\vee(x,z)$ also denote the function whose value at any point $(x,z) \in \Omega \times \mathbb{R}^n$ is the value of the distribution/function $p_2^\vee(x,z)$ at the point z (a technical point). By similar logic as in the second to last sentence, we know that $p_2^\vee \in C^\infty(\Omega \times \mathbb{R}^n)$.

Now, I claim that for any compact subset $K \subseteq \Omega$ and any $\alpha, \beta, \gamma \in \mathcal{I}(n)$,

$$(5.2) \qquad \sup_{(x,z)\in K\times\mathbb{R}^n} \left| z^\gamma D_z^\beta D_x^\alpha p_2^\vee(x,z) \right| < \infty.$$

We prove this in cases:

Cases $|\gamma| > m + n + |\alpha| + |\beta|$: We prove these cases by induction on the size of $|\beta|$. The base case of $|\beta| = 0$ and $|\gamma| > m + n + |\alpha|$ is trivial as it follows immediately from Theorem 8.8. Now suppose that $|\beta| > 0$ and $|\gamma| > m + n + |\alpha| + |\beta|$. Then we have by the product rule that

$$D_z^\beta D_x^\alpha f_\gamma(x,z) = z^\gamma D_z^\beta D_x^\alpha p_2^\vee(x,z) + \sum_{\substack{0<\omega\leq\beta \\ \omega\leq\gamma}} \frac{\beta!}{\omega!\,(\beta-\omega)!} \frac{\gamma!}{(\gamma-\omega)!} z^{\gamma-\omega} D_z^{\beta-\omega} D_x^\alpha p_2^\vee(x,z).$$

The left-hand side here is bounded over $K \times \mathbb{R}^n$ by Theorem 8.8 and each term in the Σ sum on the right is bounded over $K \times \mathbb{R}^n$ by the inductive hypothesis since each $|\beta - \omega| < |\beta|$ and

$$|\gamma| > m + n + |\alpha| + |\beta| \quad \Longrightarrow \quad |\gamma - \omega| > m + n + |\alpha| + |\beta - \omega|.$$

So the only term left "$z^\gamma D_z^\beta D_x^\alpha p_2^\vee(x,z)$" must then also be bounded over $K \times \mathbb{R}^n$. This establishes (5.2) in these cases.

Cases $|\gamma| \leq m + n + |\alpha| + |\beta|$: Let $N \in \mathbb{Z}_+$ be a positive integer such that $2N > m + n + |\alpha| + |\beta|$. Then we have that over $(x,z) \in K \times \mathbb{R}^n$

$$\left| z^\gamma D_z^\beta D_x^\alpha p_2^\vee(x,z) \right| \leq |z|^{|\gamma|} \left| D_z^\beta D_x^\alpha p_2^\vee(x,z) \right| \leq \begin{cases} \left| D_z^\beta D_x^\alpha p_2^\vee(x,z) \right| & \text{if } |z| \leq 1 \\ |z|^{2N} \left| D_z^\beta D_x^\alpha p_2^\vee(x,z) \right| & \text{if } |z| > 1 \end{cases}$$

$$= \begin{cases} \left| D_z^\beta D_x^\alpha p_2^\vee(x,z) \right| & \text{if } |z| \leq 1 \\ \left| \left(\sum_{j=1}^n z^{2e_j} \right)^N D_z^\beta D_x^\alpha p_2^\vee(x,z) \right| & \text{if } |z| > 1 \end{cases}$$

The quantity in the $|z| \leq 1$ region is bounded by continuity. For the region $z > 1$, notice that if we distribute the Σ sum we'll get lots of terms that we know are bounded by the previous case. This establishes (5.2) in these cases.

Having proved (5.2), we are ready for the last step to prove (1). In particular, notice that (5.2) implies that for all fixed $x \in \Omega$ and any $\alpha \in \mathcal{I}(n)$, $D_x^\alpha p_2^\vee(x,z)$ is a Schwartz function of z and thus the classic Fourier transform of $p_2^\vee(x,z)$ is equal to $p(x,z)$. Ok, take any compact subset $K \subseteq \Omega$ and any $\alpha, \beta, \gamma \in \mathcal{I}(n)$. Since the Fourier transform $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is continuous, there exists a constant $C > 0$ and a finite collection of indices $\{\beta_j, \gamma_j \subseteq \mathcal{I}(n) : j = 1, \ldots, M\}$ such that

$$\sup_{y\in\mathbb{R}^n}\left|y^\gamma D_y^\beta\mathcal{F}(\phi)\right| \le C\sum_{j=1}^{M}\sup_{z\in\mathbb{R}^n}\left|z^{\gamma_j}D_z^{\beta_j}\phi\right| \qquad \forall\phi\in\mathcal{S}(\mathbb{R}^n).$$

Hence, for any $(x,y)\in K\times\mathbb{R}^n$ we have that

$$\left|y^\gamma D_y^\beta D_x^\alpha p(x,y)\right| = \left|y^\gamma D_y^\beta D_x^\alpha\int e^{-2\pi i y\cdot z}p_2^\vee(x,z)dz\right| = \left|y^\gamma D_y^\beta\int e^{-2\pi i y\cdot z}D_x^\alpha p_2^\vee(x,z)dz\right|$$

$$\le C\sum_{j=1}^{M}\sup_{y\in\mathbb{R}^n}\left|z^{\gamma_j}D_z^{\beta_j}D_x^\alpha p_2^\vee(x,z)\right|,$$

which we know is finite by (5.2). It's easy to see that this implies that $p\in S^{-\infty}(\Omega)$.

Now let's prove (2). Suppose first that Ω is dense in $\mathbb{R}^n$. Then of course $\mathcal{O}$ is dense in $\Omega\times\mathbb{R}^n$. Since $p_2^\vee(x,z)$ is zero on $\mathcal{O}$, we have in fact that $p_2^\vee(x,z)$ is zero on all of $\Omega\times\mathbb{R}^n$ by continuity. Since $p(x,z)$ is the Fourier transform of $p_2^\vee(x,z)$ in the variable z, we get that indeed $p\equiv 0$. Now suppose that Ω is not dense in $\mathbb{R}^n$. Take any $\phi\in C_c^\infty(\Omega)$ and any $\psi\in C_c^\infty(\mathbb{R}^n)$ such that $\operatorname{supp}\psi$ is disjoint from Ω. Observe that if we set $p\in C^\infty(\Omega\times\mathbb{R}^n)$ to be the symbol

$$p(x,y)=\phi(x)e^{-2\pi i x\cdot y}\hat{\psi}(-y)$$

then we have that

$$p_2^\vee(x,z)=\phi(x)\psi(x-y)$$

and so

$$p_2^\vee(x,x-z)=\phi(x)\psi(z)=0.$$

Thus $p(x,D)=0$ and so p is a nonzero symbol of P.

■

5.2 Equivalent Condition for Proper Support (Proposition 8.12)

Here I work through the proof of the following theorem by putting it into my own words. One of the things that I do differently here from the book is that in the proof of the forward direction of (2), I instead consider a compact neighborhood of the set that the author calls "C." I think that this is either needed, or helps avoid a certain amount of extra work.

Theorem 5.3: *Suppose that $\Omega\subseteq\mathbb{R}^n$ is an open set. A linear map $T:C_c^\infty(\Omega)\to C^\infty(\Omega)$ is properly supported if and only if the following two conditions hold:*

1. *For any compact subset $A\subseteq\Omega$ there exists a compact subset $B\subseteq\Omega$ such that $\operatorname{supp}Tu\subseteq B$ whenever $\operatorname{supp}u\subseteq A$.*

2. *For any compact subset $A \subseteq \Omega$ there exists a compact subset $C \subseteq \Omega$ such that $Tu = 0$ on A whenever $u = 0$ on C.*

Proof: Let K denote the distribution kernel of T. First suppose that T is properly supported, or in other words: $\operatorname{supp} K$ is a proper subset of $\Omega \times \Omega$. Let's start by proving that (1) holds. Take any compact subset $A \subseteq \Omega$. Let $B \subseteq \Omega$ be the compact subset

$$B = \pi_x[(\Omega \times A) \cap \operatorname{supp} K].$$

Observe that

$$\operatorname{supp} K \cap (B^c \times A) = \emptyset.$$

Now, take any $u \in C_c^\infty(\Omega)$ such that $\operatorname{supp} u \subseteq A$. By the above equation we have that for any $v \in C_c^\infty(\Omega)$ such that $\operatorname{supp} v \subseteq B^c$,

$$\langle Tu, v\rangle = \langle K, v \otimes u\rangle = 0.$$

Thus $\operatorname{supp} Tu \subseteq B$ and so (1) holds.

Now let's prove that (2) holds. Take any compact subset $A \subseteq \Omega$. Let $A' \subseteq \Omega$ be a compact neighborhood of A in Ω. Let $C' \subseteq \Omega$ be the compact subset

$$C' = \pi_y[(A' \times \Omega) \cap \operatorname{supp} K].$$

Let $C \subseteq \Omega$ be a compact neighborhood of C' in Ω. Observe that as before we have that

$$\operatorname{supp} K \cap (A' \times (C')^c) = \emptyset.$$

Now, take any $u \in C_c^\infty(\Omega)$ such that $u = 0$ on C. Then $\operatorname{supp} u \subseteq (C')^c$. Notice that for any $v \in C_c^\infty(\Omega)$ such that $\operatorname{supp} v \subseteq (A')^{\text{int}}$, the above equation implies that

$$\langle Tu, v\rangle = \langle K, v \otimes u\rangle = 0.$$

Hence $Tu = 0$ on $(A')^{\text{int}}$ and thus on A as well. So (2) holds.

Now suppose that conditions (1) and (2) hold. Take any compact subset $A \subseteq \Omega$. First let's show that $\operatorname{supp} K \cap (\Omega \times A)$ is compact in $\Omega \times \Omega$. Let $A' \subseteq \Omega$ be a compact neighborhood of A in Ω. By (1) there exists a compact subset $B' \subseteq \Omega$ such that $\operatorname{supp} Tu \subseteq B'$ whenever $\operatorname{supp} u \subseteq A'$. Observe that for any $u, v \in C_c^\infty(\Omega)$ such that $\operatorname{supp} u \subseteq (A')^{\text{int}}$ and $\operatorname{supp} v \subseteq (B')^c$, we have that $\operatorname{supp} Tu \subseteq B'$ and so

$$\langle K, v \otimes u\rangle = \langle Tu, v\rangle = 0.$$

Thus

$$\operatorname{supp} K \cap \left((B')^c \times (A')^{\text{int}}\right) = \emptyset$$

and so

$$\operatorname{supp} K \cap (\Omega \times A) \subseteq B' \times A.$$

Since the left-hand side here is closed and the right-hand side is compact, this implies that $\operatorname{supp} K \cap (\Omega \times A)$ is compact.

Now let's show that $\operatorname{supp} K \cap (A \times \Omega)$ is compact. Letting A' be as before, by (2) we have that there exists a compact subset $C' \subseteq \Omega$ such that $Tu = 0$ on A' whenever $u = 0$ on C'. Observe that if $u, v \in C_c^\infty(\Omega)$ are such that $\operatorname{supp} u \subseteq (C')^c$ and $\operatorname{supp} v \subseteq (A')^{\text{int}}$, then $u = 0$ on C', thus $Tu = 0$ on A', and so

$$\langle K, v \otimes u \rangle = \langle Tu, v \rangle = 0.$$

Hence

$$\operatorname{supp} K \cap \left((A')^{\text{int}} \times (C')^c\right) = \emptyset$$

and so

$$\operatorname{supp} K \cap (A \times \Omega) \subseteq A \times C'.$$

Again, since the left-hand side here is closed and the right-hand side is compact, this implies that $\operatorname{supp} K \cap (A \times \Omega)$ is compact. Thus $\operatorname{supp} K$ is a proper subset of $\Omega \times \Omega$ and so T is indeed properly supported.

■

5.3 A Useful Lemma for Amplitudes

For a reason that might not be clear at first, the following lemma and its variants turn out to be extremely useful in the theory of amplitudes. One of their most common applications is to justify interchanging limits and partials with the outer integral in the definition of the amplitude maps P_a for $a \in A^m(\Omega)$. For instance, this lemma can be used to prove that the maps P_a indeed map $C_c^\infty(\Omega)$ into $C^\infty(\Omega)$ and that the mapping is continuous.

Lemma 5.4: *Suppose that $a \in A^m(\Omega)$ is an amplitude. Suppose also that $B \subseteq \Omega$ is a compact subset, $\alpha, \beta, \gamma \in \mathcal{I}(n)$ are multi-indices, and that $N \in \mathbb{Z}_+$ is a positive integer. Then there exists a positive constant $C_{B,\alpha,\beta,\gamma,N} > 0$ such that*

$$\left| \int e^{2\pi i (x-y)\cdot\xi} D_x^\beta D_y^\gamma D_\xi^\alpha a(x,\xi,y) u(y) dy \right| \leq C_{B,\alpha,\beta,\gamma,N} (1 + |\xi|)^{m-|\alpha|-2N} \sum_{|\sigma|\leq 2N} \sup D^\sigma u$$

$$\forall u \in C_c^\infty(\Omega) : \operatorname{supp} u \subseteq B \quad \textit{and} \quad \forall x \in B.$$

Remark: The reason for having "$2N$" in the estimate is that it makes the proof easier below. In particular, the convenience is that we can expand the quantity $|\xi|^{2N}$ in a simple manner using the distributive property.

Proof: Let $\{C_{N,\sigma} : \sigma \in \mathcal{I}(n) \text{ such that } |\sigma| \leq 2N\}$ be the constants in the expression

$$|\xi|^{2N} = \sum_{|\sigma|\leq 2N} C_{N,\sigma}\xi^{\gamma}.$$

Then, for any $u \in C_c^\infty(\Omega)$ such that $\operatorname{supp} u \subseteq B$ and any $x \in B$ we have that (in the third equality below I integrate by parts in y)

$$|\xi|^{2N}\left|\iint e^{2\pi i(x-y)\cdot\xi} D_x^\beta D_y^\gamma D_\xi^\alpha a(x,\xi,y)u(y)dy\right|$$

$$= \left|\sum_{|\sigma|\leq 2N} C_{N,\sigma}\int \xi^\sigma e^{-2\pi i y\cdot\xi} D_x^\beta D_y^\gamma D_\xi^\alpha a(x,\xi,y)u(y)dy\right|$$

$$= \left|\sum_{|\sigma|\leq 2N} C_{N,\sigma}\int D_y^\sigma\left[e^{-2\pi i y\cdot\xi}\right] D_x^\beta D_y^\gamma D_\xi^\alpha a(x,\xi,y)u(y)dy\right|$$

$$= \left|\sum_{|\sigma|\leq 2N} C_{N,\sigma}\int e^{-2\pi i y\cdot\xi} D_y^\sigma\left[D_x^\beta D_y^\gamma D_\xi^\alpha a(x,\xi,y)u(y)\right]dy\right|$$

Applying the product rule to the D_y^σ partial in the last quantity and then appealing to Definition 2.24, it's not hard to see that there exists a constant $C_{B,\alpha,\beta,\gamma,N} > 0$ such that the above quantity is less than or equal to

$$C_{B,\alpha,\beta,\gamma,N}(1+|\xi|)^{m-|\alpha|}\sum_{|\sigma|\leq 2N}\sup D^\sigma u$$

Dividing through by $|\xi|^{2N}$ gives

$$\left|\iint e^{2\pi i(x-y)\cdot\xi} D_x^\beta D_y^\gamma D_\xi^\alpha a(x,\xi,y)u(y)dy\right| \leq C_{B,\alpha,\beta,\gamma,N}|\xi|^{-2N}(1+|\xi|)^{m-|\alpha|}\sum_{|\sigma|\leq 2N}\sup D^\sigma u.$$

Since the left-hand side is continuous in x and ξ, from here we see that we can increase the value of $C_{B,\alpha,\beta,\gamma,N} > 0$ to make it so that the inequality in the statement of the lemma holds. This proves the lemma.

■

A useful variant of the above lemma that is proved in almost exactly the same way is the following.

Lemma 5.5: *Suppose that $a \in A^m(\Omega)$ is an amplitude. Suppose also that $B \subseteq \Omega$ is a compact subset and that the set*

$$A = \{y \in \Omega : a(x,\xi,y) \neq 0 \textit{ for some } (x,\xi) \in B \times \mathbb{R}^n\}$$

is compact. Let $\alpha,\beta,\gamma \in \mathcal{I}(n)$ be multi-indices and $N \in \mathbb{Z}_+$ be a positive integer. Then there exists a positive constant $C_{B,\alpha,\beta,\gamma,N} > 0$ such that

$$\left|\int e^{2\pi i(x-y)\cdot\xi} D_x^\beta D_y^\gamma D_\xi^\alpha a(x,\xi,y)u(y)dy\right| \le C_{B,\alpha,\beta,\gamma,N}(1+|\xi|)^{m-|\alpha|-2N} \sum_{|\sigma|\le 2N} \sup_A D^\sigma u$$

$$\forall u \in C^\infty(\Omega) \quad and \quad \forall x \in B.$$

5.4 Amplitude Kernels are Smooth Away from the Diagonal

In this note I prove the following fact that the author leaves to the reader to work out. It's a direct generalization of the same fact that holds for the usual symbols $S^m(\Omega)$ proved in Proposition 8.8(b) in the book. I avoid using equation (8.23) in the book; I'll discuss its rigorous meaning at the end.

Theorem 5.6: *Suppose that $\Omega \subseteq \mathbb{R}^n$ is an open set and that $a \in A^m(\Omega)$ is an amplitude. Let K denote the distribution kernel of P_a. Let $j \in \mathbb{Z}_+ \cup \{0\}$ be any nonnegative integer. Then for any multi-index $\alpha \in \mathcal{I}(n)$ such that $|\alpha| > m+n+j$, the distribution $(x-y)^\alpha K$ is of class C^j. In particular, this shows that K is C^∞ away from the diagonal $\Delta_\Omega = \{(x,y) \in \Omega\times\Omega : x = y\}$.*

Proof: Let $\alpha \in \mathcal{I}(n)$ be any such multi-index and let $\beta,\gamma \in \mathcal{I}(n)$ be multi-indices such that $|\beta|+|\gamma| \le j$. Let $\rho : \mathbb{R}^n \to \mathbb{R}$ be a smooth bump function such that $\rho \equiv 1$ on the ball $B_1(0)$ and $\rho \equiv 0$ on $[B_2(0)]^c$. Consider the sequence of smooth bump functions $\{\rho_k \in C_c^\infty(\mathbb{R}^n)\}_{k=1}^\infty$ given by $\rho_k(\xi) = \rho(\xi/k)$. Now, we have that for any test function $w \in C_c^\infty(\Omega\times\Omega)$

$$\langle D_x^\beta D_y^\gamma[(x-y)^\alpha K(x,y)], w(x,y)\rangle = (-1)^{|\beta|+|\gamma|}\langle K(x,y),(x-y)^\alpha D_x^\beta D_y^\gamma w(x,y)\rangle$$

$$= (-1)^{|\beta|+|\gamma|}\iiint e^{2\pi i(x-y)\cdot\xi} a(x,\xi,y)(x-y)^\alpha D_x^\beta D_y^\gamma w(x,y)dyd\xi dx.$$

Using a slight variant of Lemma 5.4 (i.e. change u to w there) and the dominated convergence theorem it's not hard to see that this is equal to

$$(-1)^{|\beta|+|\gamma|}\lim_{k\to\infty}\iiint e^{2\pi i(x-y)\cdot\xi}\rho_k(\xi)a(x,\xi,y)(x-y)^\alpha D_x^\beta D_y^\gamma w(x,y)dyd\xi dx.$$

The great thing about this is that now the integrand is compactly supported in the full (y,ξ,x) space and so we can apply Fubini's theorem. Let a_k denote the function $a_k(x,\xi,y) = \rho_k(\xi)a(x,\xi,y)$. Then, letting $\mathcal{F}^{-1}$ denote the ordinary (integral) inverse Fourier transform we have that the above quantity is further equal to

$$(-1)^{|\beta|+|\gamma|}\lim_{k\to\infty}\iint \mathcal{F}_\xi^{-1}\left[(-1)^{|\alpha|}D_\xi^\alpha a_k(x,\xi,y)\right]\Big|_{(x-y)} D_x^\beta D_y^\gamma w(x,y)dydx$$

$$= (-1)^{|\alpha|+|\gamma|}\lim_{k\to\infty}\iint \mathcal{F}_\xi^{-1}\left[\xi^{\beta+\gamma}D_\xi^\alpha a_k(x,\xi,y)\right]\Big|_{(x-y)} w(x,y)dydx,$$

where in the last inequality I did an integration by parts. Now I claim that this is equal to

$$(-1)^{|\alpha|+|\gamma|} \iint \mathcal{F}_{\xi}^{-1}\left[\xi^{\beta+\gamma} D_{\xi}^{\alpha} a(x,\xi,y)\right]\Big|_{(x-y)} w(x,y)dydx,$$

where notice that the inverse Fourier transform here is well defined and continuous since by assumption $\xi^{\beta+\gamma} D_{\xi}^{\alpha} a$ is integrable in ξ. This will then show that

$$D_x^{\beta} D_y^{\gamma}[(x-y)^{\alpha} K(x,y)] = (-1)^{|\alpha|+|\gamma|} \mathcal{F}_{\xi}^{-1}\left[\xi^{\beta+\gamma} D_{\xi}^{\alpha} a(x,\xi,y)\right]\Big|_{(x-y)}$$

and thus $(x-y)^{\alpha} K(x,y)$ is indeed of class C^j. Ok, we will prove my claim by showing that

$$\sup_{x,y \in B} \left\| \xi^{\beta+\gamma} D_{\xi}^{\alpha}[a_k - a](x,\xi,y) \right\|_{L^1(\xi)} \to 0$$

where $B = \operatorname{supp} w$. Observe that for any $x, y \in B$

$$\left| \xi^{\beta+\gamma} D_{\xi}^{\alpha}[a_k - a](x,\xi,y) \right| = \begin{cases} 0 & \text{if } |\xi| \le k \\ \left| \xi^{\beta+\gamma} D_{\xi}^{\alpha} a(x,\xi,y) \right| & \text{if } |\xi| \ge 2k \end{cases}$$

$$\le C_{\alpha,\beta,\gamma,B} |\xi|^{|\beta|+|\gamma|} (1+|\xi|)^{m-|\alpha|} \quad \text{if } |\xi| \le k \text{ or } |\xi| \ge 2k$$

for some constant $C_{\alpha,\beta,\gamma,B} > 0$. On the region $k < |\xi| < 2k$ we have that (here we still assume $x, y \in B$)

$$\left| \xi^{\beta+\gamma} D_{\xi}^{\alpha}[a_k - a](x,\xi,y) \right| = \left| \xi^{\beta+\gamma} D_{\xi}^{\alpha}[(\rho_k(\xi) - 1) a(x,\xi,y)] \right|$$

$$\le |\xi|^{|\beta|+|\gamma|} \sum_{\sigma \le \alpha} \binom{\alpha}{\sigma} \sup D^{\sigma}[\rho_k - 1] \left| D_{\xi}^{\alpha-\sigma} a(x,\xi,y) \right|$$

$$\le \frac{1}{k^{|\beta|+|\gamma|}} \sum_{\sigma \le \alpha} \binom{\alpha}{\sigma} \sup D^{\sigma}[\rho - 1] \frac{1}{k^{|\sigma|}} C_{\alpha-\sigma,B} \begin{cases} (1+2k)^{m-|\alpha-\sigma|} & \text{if } m - |\alpha-\sigma| \ge 0 \\ (1+k)^{m-|\alpha-\sigma|} & \text{if } m - |\alpha-\sigma| < 0 \end{cases}$$

for some constants $C_{\alpha-\sigma,B}$. Since the above bound over the region $k < |\xi| < 2k$ decays like $1/k^{|\beta|+|\gamma|-m+|\alpha|}$, there exists a constant $C > 0$ such that

$$\left| \xi^{\beta+\gamma} D_{\xi}^{\alpha}[a_k - a](x,\xi,y) \right| \le \frac{C}{k^{|\beta|+|\gamma|-m+|\alpha|}} \quad \text{if } k < |\xi| < 2k.$$

Thus for any $x, y \in B$ we have that (here $m_{\mathcal{L}}$ is the Lebesgue measure)

$$\left\| \xi^{\beta+\gamma} D_{\xi}^{\alpha}[a_k - a](x,\xi,y) \right\|_{L^1(\xi)}$$

$$\le \frac{C}{k^{|\beta|+|\gamma|-m+|\alpha|}} m_{\mathcal{L}}[B_{2k}(0) \setminus B_k(0)] + \int_{[B_{2k}(0)]^c} C_{\alpha,\beta,\gamma,B} |\xi|^{|\beta|+|\gamma|} (1+|\xi|)^{m-|\alpha|} d\xi.$$

Since the last quantity goes to zero as $k \to \infty$, this proves my claim and hence the theorem.

∎

As the author mentions in the book, if $a \in A^m(\Omega)$ then the distribution kernel K of the map $P_a : C_c^\infty(\Omega) \to C^\infty(\Omega)$ is given by

$$\langle K, w\rangle = \iiint e^{2\pi i(x-y)\cdot\xi} a(x,\xi,y)w(x,y)dyd\xi dx \qquad \forall w \in C_c^\infty(\Omega\times\Omega).$$

If one could formally interchange the inside dy and $d\xi$ integrals, then one would seem to get that $K = a_2^\vee(x, x-y, y)$ where $a_2^\vee$ denotes the inverse Fourier transform of a in the second variable. It is for the reason that we often use $a_2^\vee(x, x-y, y)$ as a piece of notation to symbolize the above distribution K.

One can actually come close to interpreting the notation $a_2^\vee(x, x-y, y)$ as an actual inverse Fourier transform using limits. Indeed, let $\rho \in C_c^\infty(\mathbb{R}^n)$ be a smooth bump function such that $\rho \equiv 1$ on the ball $\{|\xi| < 1\}$ and $\rho \equiv 0$ on $\{|\xi| > 2\}$ and consider the sequence $\{\rho_k \in C_c^\infty(\mathbb{R}^n)\}_{k=1}^\infty$ given by $\rho_k(\xi) = \rho(\xi/k)$. Then if we consider the compactly supported amplitude approximations a_k given by $a_k(x,\xi,y) = \rho_k(\xi)a(x,\xi,y)$, then it's not hard to prove using similar techniques as in the above proof that for any $w \in C_c^\infty(\Omega\times\Omega)$,

$$\iiint e^{2\pi i(x-y)\cdot\xi} a(x,\xi,y)w(x,y)dyd\xi dx = \lim_{k\to\infty} \iint (a_k)_2^\vee(x, x-y, y)w(x,y)dydx,$$

where each $(a_k)_2^\vee$ is a genuine integral inverse Fourier transform of a_k in the second variable. Observe also that another way to put this is that the kernel K of the map P_a is the limit of $\{(a_k)_2^\vee\}_{k=1}^\infty$ in the sense of distributions.

5.5 Asymptotic Expansion of Amplitude Maps (Theorem 8.27)

In this note I give a different (but equivalent) presentation of the proof of the following theorem.

Theorem 5.7: *Suppose $\Omega \subseteq \mathbb{R}^n$ is an open set and that $a \in A^m(\Omega)$ is an amplitude such that $P_a : C_c^\infty(\Omega) \to C^\infty(\Omega)$ is properly supported. Let $p \in C^\infty(\Omega\times\mathbb{R}^n)$ be the smooth function*

$$p(x,\xi) = e^{-2\pi i x\cdot\xi} P_a\left(e^{2\pi i y\cdot\xi}\right)$$

(see remark below). Then $p \in S^m(\Omega)$ and $P_a = p(x,D)$. Furthermore,

$$p(x,\xi) \sim \sum_{\alpha\in\mathcal{I}(n)} \frac{1}{\alpha!}\partial_\xi^\alpha D_y^\alpha a(x,\xi,x)$$

(see remark below).

Remark: In the definition of $p(x,\xi)$, the quantity $P_a(e^{2\pi i y\cdot\xi})$ signifies P_a applied to the smooth function $y \mapsto e^{2\pi i y\cdot\xi}$. This makes sense since, as is explained in the text, properly supported (continuous) maps from $C_c^\infty(\Omega)$ to $C^\infty(\Omega)$ such as P_a extend to continuous $C^\infty(\Omega) \to C^\infty(\Omega)$ maps. Next, a precise formulation of what we mean by the above equation is that

$$p(x,\xi) \sim \sum_{j=0}^{\infty} p_j(x,\xi) \qquad \text{where each} \quad p_j(x,\xi) = \sum_{|\alpha|=j} \frac{1}{\alpha!} \partial_\xi^\alpha D_y^\alpha a(x,\xi,x).$$

Proof: By Proposition 8.26 in the book, we can modify a so that the following three things happen:

1.) The values of a are unchanged in a neighborhood of the diagonal

$$\{(x,\xi,y) \in \Omega \times \mathbb{R}^n \times \Omega : x = y\}.$$

2.) The set

$$\Sigma_a = \overline{\{(x,y) \in \Omega \times \Omega : a(x,\xi,y) \neq 0 \text{ for some } \xi \in \mathbb{R}^n\}}$$

is proper in $\Omega \times \Omega$ (this set often useful since it contains the support of the kernel of P_a).

3.) The map P_a is unchanged.

Observe that if we prove the theorem for this modified a, then the theorem will hold for the original a as well. So suppose that we made this modification to a.

Ok, fix any $x \in \Omega$. For any $u \in C_c^\infty(\Omega)$ we have that

$$P_a u(x) = \iint e^{2\pi i(x-y)\cdot\xi} a(x,\xi,y) u(y) dy d\xi = \iiint e^{2\pi i(x-y)\cdot\xi} a(x,\xi,y) e^{2\pi i y\cdot\eta} \hat{u}(\eta) d\eta dy d\xi$$

$$= \iiint e^{2\pi i(x-y)\cdot\xi} a(x,\xi,y) e^{2\pi i y\cdot\eta} dy\, \hat{u}(\eta) d\eta d\xi$$

where the switching of the $d\eta$ and dy integrals in the last step is justified since Σ_a is proper and thus the set of y such that $a(x,\xi,y) \neq 0$ for some $\xi \in \mathbb{R}^n$ is compact. For the same reason and Lemma 5.5 we have that this dy integral also decays faster than any power of $|\xi|$ and is of polynomial growth in $|\eta|$ and so it is justified to interchange the last $d\eta$ and $d\xi$ integrals to get that the above quantity is further equal to

$$\iiint e^{2\pi i(x-y)\cdot\xi} a(x,\xi,y) e^{2\pi i y\cdot\eta} dy d\xi\, \hat{u}(\eta) d\eta.$$

It's easy to see that this is equal to

$$\int P_a\left(e^{2\pi i y\cdot\eta}\right) \hat{u}(\eta) d\eta = \int e^{2\pi i x\cdot\xi} p(x,\eta) \hat{u}(\eta) d\eta.$$

If we prove that $p \in S^m(\Omega)$, then the first claim in the theorem will follow. We will in fact prove right now that the following two facts holds:

a) There exists a sequence of numbers $\{\mu_j : j = 0,1,2,\ldots\}$ such that $\mu_j \to -\infty$ and such that for any compact subset $B \subseteq \Omega$, for each $k \geq 0$

$$\sup_{x\in B}\left|p(x,\xi)-\sum_{j=0}^{k}p_j(x,\xi)\right|\le C_{B,k}(1+|\xi|)^{\mu_k}$$

for some constants $C_{B,k}>0$ where p_j are defined in the above remark.

b) For any multi-indices $\alpha,\beta\in\mathcal{I}(n)$, there exists a real number $\mu(\alpha,\beta)$ such that for any compact subset $B\subseteq\Omega$,

$$\sup_{x\in B}\left|D_x^\beta D_\xi^\alpha p(x,\xi)\right|\le C_{B,\alpha,\beta}(1+|\xi|)^{\mu(\alpha,\beta)}.$$

for some constant $C_{B,\alpha,\beta}>0$.

It's not hard to see that each $p_j\in S^{m-j}(\Omega)$ and so by Theorem 8.20 in the book we will have that (a) and (b) together show that both $p\in S^m(\Omega)$ and $p\sim\sum_{j=0}^{\infty}p_j$. Hence the theorem will be proved.

Let's start by proving (b). We have that

$$(5.8)\qquad p(x,\xi)=e^{-2\pi i x\cdot\xi}\iint e^{2\pi i(x-y)\cdot\eta}a(x,\eta,y)e^{2\pi i y\cdot\xi}dyd\eta.$$

Take any $\alpha,\beta\in\mathcal{I}(n)$. Let $B\subseteq\Omega$ be any compact subset. Let $\tilde{B}\subseteq\Omega$ be a compact neighborhood of B in Ω (i.e. we technically need fatten B a little since we'll be considering D_x partials of p at points on the boundary of B as well). Since Σ_a is proper, the set of y such that $a(x,\eta,y)\ne0$ for some $(x,\eta)\in\tilde{B}\times\mathbb{R}^n$ is compact. Thus by Lemma 5.5 we can interchange D_ξ partials with the double integrals in the above equation to get that over $x\in B$ (I apply the product rule here),

$$D_\xi^\alpha p(x,\xi)=\sum_{\gamma\le\alpha}\binom{\alpha}{\gamma}(-1)^{|\gamma|}x^\gamma e^{-2\pi i x\cdot\xi}\iint e^{2\pi i(x-y)\cdot\eta}a(x,\eta,y)y^{\alpha-\gamma}e^{2\pi i y\cdot\xi}dyd\eta.$$

By similar reasoning, we can interchange D_x partials with the double integrals in the above equation to get that over $x\in B$,

$$D_x^\beta D_\xi^\alpha p(x,\xi)$$
$$=\sum_{\gamma\le\alpha}\binom{\alpha}{\gamma}(-1)^{|\gamma|}\sum_{\sigma\le\beta}\binom{\beta}{\sigma}D_x^\sigma\left[x^\gamma e^{-2\pi i x\cdot\xi}\right]\iint D_x^{\beta-\sigma}\left[e^{2\pi i(x-y)\cdot\eta}a(x,\eta,y)\right]y^{\alpha-\gamma}e^{2\pi i y\cdot\xi}dyd\eta$$

Looking at the double integrals in the last quantity, after distributing the $D_x^{\beta-\sigma}$ partial it's not hard to see by Lemma 5.5 that there exists an $M\in\mathbb{Z}_+$ independent of B and a positive constant $C_{B,\alpha,\beta}>0$ such that the quantity in the above equation satisfies the bound

$$\left|D_x^\beta D_\xi^\alpha p(x,\xi)\right|\le C_{B,\alpha,\beta}(1+|\xi|)^M$$

(for instance, $M = |\beta| + 2N$ will work where $N > 0$ is any integer such that $|\beta| + m - 2N < -n - 1$). This proves (b).

Now let's prove (a). To help avoid possible confusion, I will sometimes write my partials as D_j where $j = 1,2,3$ indicates what argument I'm differentiating a function in. Take any compact subset $B \subseteq \Omega$. The idea here is to use Taylor's theorem on the function $\zeta \mapsto a(x, \zeta, y)$ centered at $\zeta = \xi$. We have by (5.8) that for any $j \geq 0$ (here $R_{x,y,\xi,j}$ is the Tailor series error term for the just mentioned function)

$$(5.9) \qquad p(x,\xi) = \iint e^{2\pi i(x-y)\cdot(\eta-\xi)} a(x,\eta,y) dy d\eta$$

$$= \sum_{|\alpha|\leq k} \frac{1}{\alpha!} \iint e^{2\pi i(x-y)\cdot(\eta-\xi)} \partial_2^\alpha a(x,\xi,y)(\eta-\xi)^\alpha dy d\eta + \iint e^{2\pi i(x-y)\cdot(\eta-\xi)} R_{x,y,\xi,k}(\eta) dy d\eta.$$

Let's take a look at the first sum on the right-hand side. Over $(x,\xi) \in B \times \mathbb{R}^n$ we have that every integral in that sum it is equal to

$$\iint (-1)^{|\alpha|} D_y^\alpha \left[e^{2\pi i(x-y)\cdot(\eta-\xi)}\right] \partial_2^\alpha a(x,\xi,y) dy d\eta = \iint e^{2\pi i(x-y)\cdot(\eta-\xi)} D_3^\alpha \partial_2^\alpha a(x,\xi,y) dy d\eta$$

where in the last equality I did integration by parts in y using the fact that the set of y such that $a(x,\xi,y) \neq 0$ for some $(x,\xi) \in B \times \mathbb{R}^n$ is compact. Letting $\mathcal{F}$ denote the Fourier transform, observe that the above quantity can further be written as

$$\int e^{2\pi i x\cdot(\eta-\xi)} \mathcal{F}_y[D_3^\alpha \partial_2^\alpha a(x,\xi,y)]\Big|_{(\eta-\xi)} d\eta = D_3^\alpha \partial_2^\alpha a(x,\xi,x).$$

Taking a look back at (5.9), we see that (a) will be proved then if we can show that over $(x,\xi) \in B \times \mathbb{R}^n$ there exists a sequence of numbers $\mu_j : j = 0,1,2,\ldots$ independent of B such that $\mu_j \to -\infty$ and the integral of the error term satisfies

$$(5.10) \qquad \left|\iint e^{2\pi i(x-y)\cdot(\eta-\xi)} R_{x,y,\xi,k}(\eta) dy d\eta\right| \leq C_{B,k}(1+|\xi|)^{\mu_k}$$

for some constants $C_{B,k} > 0$. Using Taylor's theorem, we can write out the integral on the left-hand side here as

$$(k+1) \sum_{|\alpha|=k+1} \frac{1}{\alpha!} \iint \int_0^1 e^{2\pi i(x-y)\cdot(\eta-\xi)} (1-t)^k \partial_2^\alpha a(x,\xi+t(\eta-\xi),y)(\eta-\xi)^\alpha dt\, dy d\eta.$$

Let's take a look at each integral in this sum. Call the integral in the α^{th} term in the above sum $I_\alpha(x,\xi)$:

$$I_\alpha(x,\xi) = \iint \int_0^1 e^{2\pi i(x-y)\cdot(\eta-\xi)} (1-t)^k \partial_2^\alpha a(x,\xi+t(\eta-\xi),y)(\eta-\xi)^\alpha dt\, dy d\eta$$

$$= \int \int_0^1 (1-t)^k \int (-1)^{|\alpha|} D_y^\alpha \left[e^{2\pi i (x-y)\cdot(\eta-\xi)} \right] \partial_2^\alpha a(x, \xi + t(\eta - \xi), y) dy\, dt\, d\eta$$

$$= \int \int_0^1 (1-t)^k \int e^{2\pi i (x-y)\cdot(\eta-\xi)} D_3^\alpha \partial_2^\alpha a(x, \xi + t(\eta - \xi), y) dy\, dt\, d\eta$$

where in the last equality I did integration by parts in y. Furthermore, let $J_\alpha(x, \xi, t, \eta)$ denote the inner dy integral in the last quantity:

$$J_\alpha(x, \xi, t, \eta) = \int e^{2\pi i (x-y)\cdot(\eta-\xi)} D_3^\alpha \partial_2^\alpha a(x, \xi + t(\eta - \xi), y) dy.$$

Over the region $|\eta - \xi| < |\xi|/2$, we see from this equation directly that there exists a constant $C_{B,k,\alpha} > 0$ such that

$$|J_\alpha(x, \xi, t, \eta)| \leq C_{B,k,\alpha} (1 + 3\,|\xi|/2)^{m-|\alpha|} = C_{B,k,\alpha} (1 + 3\,|\xi|/2)^{m-k}.$$

Over the region $|\eta - \xi| \geq |\xi|/2$, it's not hard to see by repeating similar steps as in the proof of Lemma 5.4 that for any positive integer $N \in \mathbb{Z}_+$ there exists a constant $C_{B,k,\alpha,N}$ such that

$$|J_\alpha(x, \xi, t, \eta)| \leq C_{B,k,\alpha,N} |\eta - \xi|^{-2N} (1 + |\xi + t(\eta - \xi)|)^{m-k}$$

$$\leq \begin{cases} C_{B,k,\alpha,N} |\eta - \xi|^{-2N} (1 + |\xi|)^{m-k} (1 + |\eta - \xi|)^{m-k} & \text{if } \ m - k \geq 0 \\ C_{B,k,\alpha,N} |\eta - \xi|^{-2N} & \text{if } \ m - k < 0 \end{cases}$$

(the triangle inequality was used in the $m - k \geq 0$ case). Thus in the following two cases we have that (here $m_\mathcal{L}$ is the Lebesgue measure)

i. If $m - k \geq 0$:

$$|I_\alpha(x, \xi)| \leq C_{B,k,\alpha} (1 + 3\,|\xi|/2)^{m-k} m_\mathcal{L}\{\eta : |\eta - \xi| < |\xi|/2\} + C_{B,k,\alpha,N} (1 + |\xi|)^{m-k} \int_{\eta : |\eta - \xi| \geq |\xi|/2} |\eta - \xi|^{-2N} (1 + |\eta - \xi|)^{m-k} d\eta.$$

   If we choose N big enough, the right-hand side grows no faster than $|\xi|^{m-k+n}$.

ii. If $m - k < 0$:

$$|I_\alpha(x, \xi)| \leq C_{B,k,\alpha} (1 + 3\,|\xi|/2)^{m-k} m_\mathcal{L}\{\eta : |\eta - \xi| < |\xi|/2\} + C_{B,k,\alpha,N} \int_{\eta : |\eta - \xi| \geq |\xi|/2} |\eta - \xi|^{-2N} d\eta.$$

   In this case, we also have that if we choose N big enough the right-hand side grows no faster than $|\xi|^{m-k+n}$.

From both cases we see that there exists a constant $C'_{B,\alpha} > 0$ such that

$$|I_\alpha(x,\xi)| \le C'_{B,\alpha}(1+|\xi|)^{m-k+n} \qquad \forall x \in B.$$

Plugging this back into the equation following right after (5.10), we have that there exists some constant $C_{B,k} > 0$ such that

$$\left|\iint e^{2\pi i(x-y)\cdot(\eta-\xi)} R_{x,y,\xi,k}(\eta)\,dy\,d\eta\right| \le C_{B,k}(1+|\xi|)^{m-k+n}.$$

Setting μ_k to be the sequence $\mu_k = m-k+n$, this proves (a). As discussed above, this proves the theorem.

■

Because of the above theorem, we can establish the following useful piece of notation.

Notation 5.11: *Suppose that $P \in \Psi^m(\Omega)$ is a properly supported pseudodifferential operator. We let $\sigma_P \in C^\infty(\Omega \times \mathbb{R}^n)$ denote the following function:*

$$\sigma_P(x,\xi) = e^{-2\pi i x\cdot\xi} P\left(e^{2\pi i y\cdot\xi}\right).$$

The great thing about this function σ_P is that because of the above theorem we know that $\sigma_P \in S^m(\Omega)$ and that it is a symbol of P: $P = \sigma_P(x,D)$.

5.6 Pseudodifferential Operators and Sobolev Spaces (Theorem 8.40)

Here I give a slightly different proof of Theorem 8.40 in the book, which I break up into two theorems.

Theorem 5.12: *Suppose that $P \in \Psi^m(\Omega)$ and that $s \in \mathbb{R}$. Then P maps $H_s^0(\Omega)$ continuously into $H_{s-m}^{loc}(\Omega)$ (see the remark below).*

Remark: Here is how we interpret this theorem. In the proof below we will show that P maps the subspace $C_c^\infty(\Omega) \subseteq H_s(\mathbb{R}^n)$ continuously into $H_{s-m}^{\mathrm{loc}}(\Omega)$. Precisely, this means that for any $\phi \in C_c^\infty(\Omega)$ there exists constant $C > 0$ such that for any $u \in C_c^\infty(\Omega)$,

$$\|\phi P u\|_{s-m} \le C\|u\|_s.$$

This continuity then implies that P extends uniquely to a continuous $P : H_s^0(\Omega) \to H_{s-m}^{\mathrm{loc}}(\Omega)$ map (this technically uses the completeness of $H_{s-m}^{\mathrm{loc}}(\Omega)$, see Proposition 4.5).

Proof: Pick any $\phi \in C_c^\infty(\Omega)$. As mentioned in the above remark, it will be enough to prove the existence of a constant $C > 0$ such that $\|\phi P u\|_{s-m} \le C\|u\|_s$ for all $u \in C_c^\infty(\Omega)$. Observe that if $p \in S^m(\Omega)$ is a symbol of P, then $\phi P = q(x,D)$ were $q = \phi(x)p(x,\xi)$. The symbol $q \in S^m(\Omega)$ of course satisfies the property that $q(x,\xi) = 0$ for x outside the compact set $\operatorname{supp}\phi$. Now, for any $u \in C_c^\infty(\Omega)$ we have that

$$(5.13) \qquad \widehat{\phi P u}(\eta) = \iint e^{2\pi i(\xi-\eta)\cdot x} q(x,\xi)\hat{u}(\xi)\,d\xi\,dx = \int \hat{q}_1(\eta-\xi,\xi)\hat{u}(\xi)\,d\xi,$$

where $\hat{q}_1$ denote the Fourier transform of q in the first variable. Thus, we have that

$$\|\phi P u\|_{s-m} = \left\| \int (1+|\eta|^2)^{(s-m)/2} \hat{q}_1(\eta-\xi,\xi)\hat{u}(\xi)d\xi \right\|_{L^2} = \left\| \int K(\eta,\xi) f(\xi) d\xi \right\|_{L^2}$$

where

$$K(\eta,\xi) = (1+|\eta|^2)^{(s-m)/2}(1+|\xi|^2)^{-s/2}\hat{q}_1(\eta-\xi,\xi) \quad \text{and} \quad f(\xi) = (1+|\xi|^2)^{s/2}\hat{u}(\xi).$$

Using similar techniques that we used to prove Lemma 5.4, it's not hard to see that for any $N \in \mathbb{Z}_+$ there exists a constant $C_N > 0$ such that

$$|\hat{q}_1(\zeta,\xi)| \le C_N(1+|\zeta|^2)^{-N}(1+|\xi|^2)^{m/2} \qquad \forall \zeta,\xi \in \mathbb{R}^n.$$

Thus for any $N \in \mathbb{Z}_+$, by Lemma 6.10 in the book (the little unnamed inequality)

$$|K(\eta,\xi)| \le C_N(1+|\eta|^2)^{(s-m)/2}(1+|\xi|^2)^{(m-s)/2}(1+|\eta-\xi|^2)^{-N}$$
$$\le C_N 2^{|s-m|/2}(1+|\eta-\xi|^2)^{|s-m|/2-N}.$$

So, we can let N be big enough so that K is in $L^1(\mathbb{R}^n)$ in each variable separately. Hence by Young's inequality for integral operators (Theorem 0.10 in the book) we have that

$$\|\phi P u\|_{s-m} \le C_\phi \|f\|_{L^2} = C_\phi \|u\|_s$$

for some constant $C_\phi > 0$ only dependent on ϕ. This proves the theorem.

■

If the pseudodifferential operator is properly supported, then we can say something else:

Theorem 5.14: *Suppose that $P \in \Psi^m(\Omega)$ is properly supported and that $s \in \mathbb{R}$. Then P maps $H_s^{loc}(\Omega)$ continuously into $H_{s-m}^{loc}(\Omega)$.*

Remark: Here we don't need any special interpretations since P is perfectly defined over $H_s^{\text{loc}}(\Omega) \subseteq \mathcal{D}'(\Omega)$ (see page 277 in the book).

Proof: Take any $\phi \in C_c^\infty(\Omega)$. We will prove this theorem by showing that there exists a constant $C > 0$ and a $\psi \in C_c^\infty(\Omega)$ such that $\|\phi P u\|_{s-m} \le C\|\psi u\|_s$ for all $u \in H_s^{\text{loc}}(\Omega)$. Let $A \subseteq \Omega$ be a compact subset of Ω that contains $\operatorname{supp}\phi$ in its interior. By Theorem 5.3 there exists a compact subset $C \subseteq \Omega$ such that $P\rho = 0$ on A whenever $\rho \in C_c^\infty(\Omega)$ is zero on C. Let $\psi \in C_c^\infty(\Omega)$ be such that $\psi \equiv 1$ on a neighborhood of C. Then, by the solution of Exercise 4 of Section 8.B in the book we have that $Pu|_{A^{\text{int}}} = P(\psi u)|_{A^{\text{int}}}$ for all $u \in \mathcal{D}'(\Omega)$. Hence $\phi P u = \phi P(\psi u)$ for all $u \in \mathcal{D}'(\Omega)$, and in particular for $u \in H_s^{\text{loc}}(\Omega)$. Now, as in the proof of Theorem 5.12 observe that if $p \in S^m(\Omega)$ is a symbol of P, then $\phi P = q(x,D)$ where $q(x,\xi) = \phi(x)p(x,\xi)$. Notice that the symbol $q \in S^m(\Omega)$ satisfies the property that $q(x,\xi) = 0$ for x outside the compact set $\operatorname{supp}\phi$ and $q(x,D)$ is also properly supported. Then we have that for any $u \in H_s^{\text{loc}}(\Omega)$,

$$(\phi P u)\widehat{\ }(\eta) = \big(\phi P(\psi u)\big)\widehat{\ }(\eta) = \big(q(x,D)(\psi u)\big)\widehat{\ }(\eta) = \langle q(x,D)(\psi u)(x), e^{-2\pi i \eta\cdot x}\rangle,$$

where we've used the fact that $q(x,D)(\psi u)$ is compactly supported since q is proper (c.f. page 276 in the book). To continue the calculation, let $\rho \in C_c^\infty(\Omega)$ be such that $\rho \equiv 1$ on a neighborhood of $\text{supp}[q(x,D)(\psi u)]$. By the discussion on pages 269-270 in the book, we then have that the above quantity can further be written as

$$\langle q(x,D)(\psi u)(x), \rho(x)e^{-2\pi i\eta\cdot x}\rangle = \langle \widehat{\psi u}(\xi), \int e^{2\pi i x\cdot\xi} q(x,\xi)\rho(x)e^{-2\pi i\eta\cdot x}dx\rangle$$

$$= \langle \widehat{\psi u}(\xi), \int e^{2\pi i x\cdot\xi} q(x,\xi)e^{-2\pi i\eta\cdot x}dx\rangle.$$

Now, since $\psi u \in H_s(\Omega)$ we arrive at that

$$(\phi Pu)^\frown(\eta) = \iint e^{2\pi i(\xi-\eta)\cdot\xi} q(x,\xi)\widehat{\psi u}(\xi)dxd\xi.$$

From here we can continue just as in the proof of Theorem 5.12 starting with (5.13) to get that there exists a constant $C_\phi > 0$ only dependent on ϕ such that

$$\|\phi Pu\|_{s-m} \le C_\phi \|\psi u\|_s.$$

This proves the theorem.

■

5.7 Local Solvability of Properly Supported Elliptic Pseudodifferential Operators

Here I prove the following theorem by putting the proof of Theorem 8.45 in the book into my own words. I'm going to try to be very careful in my notation.

Theorem 5.15: *Suppose that $P \in \Psi^m(\Omega)$ is a properly supported elliptic pseudodifferential operator of order m. Pick any $f \in \mathcal{D}'(\Omega)$. Then, for any $x_0 \in \Omega$ there exists an open neighborhood U of x_0 contained in Ω such that the equation $Pu = f$ has a solution $u \in \mathcal{D}'(\Omega)$ over U (i.e. $(Pu)|_U = f|_U$).*

Proof: Let Q be a parametrix of P and consider the operator $S = (PQ - I) \in \Psi^{-\infty}(\Omega)$. Let W be an open neighborhood of x_0 such that $\overline{W} \subseteq \Omega$ is compact. Since S is properly supported, by Theorem 5.3 there exists a compact subset $A \subseteq \Omega$ such that values of Su on $\overline{W}$ only depend on the values of $u \in C_c^\infty(\Omega)$ on A. Let $\phi, \psi \in C_c^\infty(\Omega)$ be such that

$$\phi \equiv 1 \ \text{ on } \overline{W} \qquad \text{and} \qquad \psi \equiv 1 \ \text{ on } A.$$

Now, consider the operator $\tilde{T} : C_c^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ given by (see Notation 2.12 for "$\cdot^{\mathbb{R}^n}$")

$$\tilde{T}u = [\phi S(\psi \cdot [u|_\Omega])]^{\mathbb{R}^n}.$$

I put "$\sim$" over my T here to remind us that this operator is acting over $\mathbb{R}^n$ and not Ω anymore. The reason we will want to work with an operator over $\mathbb{R}^n$ is so that we can utilize $H_s(\mathbb{R}^n)$

Hilbert spaces theory. It's not hard to show that $\tilde{T}$ is a pseudodifferential operator of order $-\infty$ as well, and using Proposition 8.12 is also properly supported. Let's also observe an almost obvious technical point:

Claim: For any $v \in \mathcal{D}'(\mathbb{R}^n)$,

$$\left.\left(\tilde{T}v\right)\right|_{\Omega} = \phi S(\psi[v|_{\Omega}]).$$

Proof: We have that this already holds for $v \in C_c^\infty(\mathbb{R}^n)$ by definition. To prove it for more general distributions, we argue by continuity. Take any $v \in \mathcal{D}'(\mathbb{R}^n)$ and let $\{v_k\}_{k=1}^{\infty} \subseteq C_c^\infty(\mathbb{R}^n)$ be such that $v_k \to v$ in $\mathcal{D}'(\mathbb{R}^n)$. Then, we have that

$$\left.\left(\tilde{T}v\right)\right|_{\Omega} = \lim_{k\to\infty} \left.\left(\tilde{T}v_k\right)\right|_{\Omega} = \lim_{k\to\infty} \phi S(\psi[v_k|_{\Omega}]) = \phi S(\psi[v|_{\Omega}]).$$

//

Back to proving our theorem. Consider $\tilde{f} = (\phi f)^{\mathbb{R}^n} \in \mathcal{E}'(\mathbb{R}^n)$. We claim that if we can find a $v \in \mathcal{D}'(\mathbb{R}^n)$ such that $\left(\tilde{T} + I\right)v = \tilde{f}$ over some open neighborhood $U \subseteq W$ of x_0, then $u = Q(v|_{\Omega})$ will be the solution we're seeking. To see why, observe that

$$PQ(v|_{\Omega})|_U = (S + I)(v|_{\Omega})|_U = \left.\left[\phi S\left(\psi(v|_{\Omega})\right)\right]\right|_U + v|_U = \left.\left(\tilde{T}v + v\right)\right|_U$$

$$= \left.\tilde{f}\right|_U = f|_U.$$

Hence indeed it suffices to find a such a $v \in \mathcal{D}'(\mathbb{R}^n)$.

Since $\tilde{f} \in \mathcal{E}'(\mathbb{R}^n)$, we have that $\tilde{f} \in H_s(\mathbb{R}^n)$ for some $s \in \mathbb{R}$. Fixing such an $s \in \mathbb{R}$, the following claim is the key to applying Hilbert space theory to our problem.

Claim: The operator $\tilde{T} : H_s(\mathbb{R}^n) \to H_s(\mathbb{R}^n)$ is a compact operator.

Proof: Take any bounded subset B of $H_s(\mathbb{R}^n)$. We'll show that $\tilde{T}[B]$ has a convergent subsequence in $H_s(\mathbb{R}^n)$. Let $\rho \in C_c^\infty(\mathbb{R}^n)$ be such that $\operatorname{supp} \rho \subseteq \Omega$ and $\rho \equiv 1$ on $\operatorname{supp} \phi$. It's not hard to see by definition that $\tilde{T}v = \rho\tilde{T}(v)$ for all $v \in \mathcal{D}'(\mathbb{R}^n)$ and so we have that

$$\tilde{T}[B] = \{\rho\tilde{T}(v) : v \in B\}.$$

Now, take any $r < 0$ and recall that $\tilde{T} \in \Psi^r(\mathbb{R}^n)$. By Theorem 5.14 we have that there exists a constant $C_\rho > 0$ and a $\sigma \in C_c^\infty(\mathbb{R}^n)$ such that for any $v \in H_s(\mathbb{R}^n)$,

$$\left\|\rho\tilde{T}(v)\right\|_{s-r} \leq C_\phi \|\sigma v\|_s.$$

Since multiplication by σ is a continuous operation in $H_s(\mathbb{R}^n)$, this implies that the set in the previous equation is bounded in $H_{s-r}(\mathbb{R}^n)$. Furthermore, every member of that set is also supported in the fixed compact set $\operatorname{supp} \rho$. Hence by the proof of Rellich's theorem (Theorem 6.14 in the book) we have that this set has a subsequence that converges in $H_s(\mathbb{R}^n)$. Thus the map $\tilde{T} : H_s(\mathbb{R}^n) \to H_s(\mathbb{R}^n)$ is indeed compact.

//

Back to proving the theorem. Let $\tilde{T}_s^*$ denote the $\langle \cdot | \cdot \rangle_s$ adjoint of $\tilde{T}$ and let $\tilde{T}^*$ denote the ordinary adjoint of $\tilde{T}$ (see Definition 2.28 for the latter). By Fredholm's theorem and its corollary (Corollary 0.42 in the book) we have that $(\tilde{T}+I)v = \tilde{f}$ has a solution if and only if $\tilde{f} \perp \{g \in H_s(\mathbb{R}^n) : \tilde{T}_s^* g = -g\}$. We will use the following claim to put this condition into a more convenient form.

<u>Claim:</u> The following is true and makes sense:

$$\langle \tilde{f} | g \rangle_s = -\langle \tilde{f}, \overline{\tilde{T}^* \Lambda^{2s} g} \rangle \qquad \forall g \in H_s(\mathbb{R}^n) : T_s^* g = -g.$$

<u>Proof:</u> For shorthand, let $\mathcal{R}$ denote the set of all g considered here. First let's prove the following version of the above equation:

$$(5.16) \qquad \langle h | g \rangle_s = -\langle h, \overline{\tilde{T}^* \Lambda^{2s} g} \rangle \qquad \forall h \in C_c^\infty(\mathbb{R}^n) \ \ \forall g \in \mathcal{R}.$$

We have that for any $h \in C_c^\infty(\mathbb{R}^n)$ and any $g \in \mathcal{R}$,

$$-\overline{\langle h | g \rangle_s} = \langle \tilde{T}_s^* g | h \rangle_s = \langle g | \tilde{T} h \rangle_s = \int (1 + |\xi|^2)^s \hat{g}(\xi) \overline{\widehat{\tilde{T} h}}(\xi) d\xi.$$

Now, since h is compactly supported and $\tilde{T}$ is properly supported we have that $\tilde{T}h \in C_c^\infty(\mathbb{R}^n)$ (alternatively $\tilde{T}$ always vanishes outside of $\operatorname{supp} \phi$). So we can continue the above calculation as

$$= \langle (\Lambda^{2s} g)(-x), \left(\overline{\widehat{\tilde{T} h}}\right)^\vee (x) \rangle = \langle \Lambda^{2s} g, \overline{\tilde{T} h} \rangle.$$

Now, by definition we have that $\langle \tilde{T}^* u, \overline{v} \rangle = \langle u, \overline{\tilde{T} v} \rangle$ for all $u, v \in C_c^\infty(\mathbb{R}^n)$. By the continuity of $\tilde{T}^* : \mathcal{D}'(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^n)$ (Exercise 8.B.4 in the book) we have that this also holds for $u \in \mathcal{D}'(\mathbb{R}^n)$. So we can further continue the above calculation as

$$= \langle \tilde{T}^* \Lambda^{2s} g, \overline{h} \rangle = \overline{\langle \overline{\tilde{T}^* \Lambda^{2s} g}, h \rangle} = \overline{\langle h, \overline{\tilde{T}^* \Lambda^{2s} g} \rangle},$$

where observe that the last expression makes sense since $\tilde{T}^*$ maps $\mathcal{D}'(\mathbb{R}^n)$ maps into $C^\infty(\mathbb{R}^n)$ since $\tilde{T}^*$ is properly supported and in $\Psi^{-\infty}(\mathbb{R}^n)$ (see pages 290 – 291 in the book). Hence we've shown that

$$-\overline{\langle h | g \rangle_s} = \overline{\langle h, \overline{\tilde{T}^* \Lambda^{2s} g} \rangle},$$

which evidently implies (5.16).

Now, using the density of $C_c^\infty(\mathbb{R}^n)$ in $H_s(\mathbb{R}^n)$ and the fact that convergence in $H_s(\mathbb{R}^n)$ also implies convergence in $\mathcal{D}'(\mathbb{R}^n)$ (see Lemma 4.1), we get that (5.16) also holds if we replace h with $\tilde{f}$. This proves the claim.

//

Back to proving our theorem. Because of the above claim, we now have that $(\tilde{T}+I)v=\tilde{f}$ has a solution if and only if

$$\langle \tilde{f}, h\rangle = 0 \qquad \forall h \in \mathcal{X}$$

where

$$\mathcal{X} = \left\{\overline{\tilde{T}^*\Lambda^{2s} g} : g \in H_s(\mathbb{R}^n) \text{ such that } \tilde{T}_s^* g = -g\right\}$$

If we let $\rho \in C_c^\infty(\mathbb{R}^n) : \operatorname{supp}\rho \subseteq \Omega$ be such that $\rho \equiv 1$ on $\operatorname{supp}\tilde{f}$, then we can harmlessly change the above condition to[1]

$$\langle \tilde{f}, h\rangle = 0 \qquad \forall h \in \rho\mathcal{X}.$$

This condition might not actually not hold for $\tilde{f}$, but we can modify $\tilde{f}$ slightly so that this does. By Fredholm's theorem (Theorem 0.38 in the book) we have that $\rho\mathcal{X}$ is a finite dimensional vector space. Thus by (the technical) Lemma 8.44 in the book there exists an $\varepsilon > 0$ and a $g \in C_c^\infty(\mathbb{R}^n)$ such that g vanishes on $B_\varepsilon(x_0)$ and $\langle f-g, h\rangle = 0$ for all $h \in \rho\mathcal{X}$. Then we get that there exists a solution $v \in \mathcal{D}'(\mathbb{R}^n)$ to $(\tilde{T}+I)v = \tilde{f}-g$. Such a v solves $(\tilde{T}+I)v=\tilde{f}$ over $U = B_\varepsilon(x_0)\cap W$. As discussed before this proves the theorem.

∎

5.8 Equivalent Definition of Wavefront Set (Theorem 8.56)

In this note I put the proof of the following theorem into my own words by filling in some of the details in the proof.

Note: If you are confused as to why the following is not a definition, please see Definition 2.36 above.

Theorem 5.17: *Suppose that $u \in \mathcal{D}'(\Omega)$ is a distribution where $\Omega \subseteq \mathbb{R}^n$. Then $(x_0,\xi_0) \notin WF(u)$ if and only if there exists a $\phi \in C_c^\infty(\Omega)$ such that $\phi(x_0) \neq 0$ and an open cone $V \subseteq \mathbb{R}^n \setminus \{0\}$ containing ξ_0 such that*

$$\forall M > 0 \quad \exists C > 0 \quad \forall \xi \in V, \quad \left|\widehat{\phi u}(\xi)\right| \le C(1+|\xi|)^{-M}.$$

Proof: Take any $(x_0,\xi_0) \in T^0\Omega$. First suppose that it satisfies the final condition in the theorem with some $\phi \in C_c^\infty(\Omega)$ and $V \subseteq \mathbb{R}^n \setminus \{0\}$. We want to show that $(x_0,\xi_0) \notin WF(u)$. We leave it to the reader to show that there exists a $p \in C^\infty(\mathbb{R}^n)$ such that $p(\xi)$ is homogeneous of degree 0 for large ξ, $p(\xi_0) \neq 0$, and $\operatorname{supp} p \subseteq V$ (Folland constructs such a function in his proof of Proposition 8.52 in the book). Notice then that $p \cdot \widehat{\phi u}$ is rapidly decreasing in all directions. Hence it's a Schwartz function and so it's inverse Fourier transform is smooth. The reason the

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[1] The reason for ρ is to make the vector space $\rho\mathcal{X}$ that we're considering a subspace of $C_c^\infty(\Omega)$ (rather than just $C^\infty(\Omega)$). This is in fact unnecessary because all members of $\mathcal{X}$ already have compact support, but I'm too lazy to prove this. It follows from $T^*v = \overline{\psi}S^*(\overline{\phi}v)$ and then an extension argument to $\mathbb{R}^n$.

inverse Fourier transform is of interest that it can be written as a pseudodifferential operator (see the claim below). First observe that by Theorem 8.8(a) in the book we have that $\check{p}$ agrees with a Schwartz function away from zero.

Claim: For any $v \in \mathcal{E}'(\Omega)$,

$$p(D)[v] = \check{p} * v = \int e^{2\pi i x\cdot\xi} p(\xi)\hat{v}(\xi)d\xi.$$

where $p(D) \in \Psi^0(\Omega)$ is the operator obtained from the symbol p thought of as being held constant in the x-variable (i.e $p(x,\xi) = p(\xi)$).

Note: The reason the last expression doesn't follow immediately from the definition of pseudodifferential operators is that with no assumptions this integral representation for $p(D)$ only works when $v \in C_c^\infty(\Omega)$.

Proof: Let's start by proving the first equality. If $v \in C_c^\infty(\Omega)$, then this simply follows from the convolution theorem since p is Schwartz and hence in $\mathcal{S}'(\mathbb{R}^n)$ (see chapter 8 of Friedlander and Joshi's book on distribution theory). For general $v \in \mathcal{E}'(\Omega)$, let $\{v_k : k \in \mathbb{Z}_+\}$ be a sequence in $C_c^\infty(\Omega)$ that converges to v in $\mathcal{E}'(\Omega)$. We have already established that each $p(D)[v_k] = \check{p} * v_k$, and so passing to the limit and using the continuity of $p(D)[\dots]$ and $\check{p} * (\dots)$ establishes the equality for $v \in \mathcal{E}'(\Omega)$ as well. The second equality in the claim simply follows from the convolution theorem.

//

Back to proving our theorem. We thus have that

$$(5.18) \qquad p(D)[\phi u] \in C^\infty(\Omega)$$

Consider the operator $P \in \Psi^0(\Omega)$ given by $Pv = p(D)[\phi v]$ for $v \in C_c^\infty(\Omega)$. Notice that the symbol of P is $p(\xi)\phi(x) \pmod{S^{-1}(\Omega)}$ and hence $(x_0, \xi_0) \notin \operatorname{char}_0 P$ by construction.

Now, by Corollary 8.32 there exists a properly supported operator $\tilde{P} \in \Psi^0(\Omega)$ such that $\tilde{P} = P + R$ where $R \in \Psi^{-\infty}(\Omega)$.[2] I claim that $\tilde{P}u \in C^\infty(\Omega)$. To see why, take any precompact $U \subseteq \Omega$ and let $C \subseteq \Omega$ be such that $\tilde{P}v \equiv 0$ on $\overline{U}$ if $v \equiv 0$ on C (c.f. Proposition 8.12 in the book). Let $\psi \in C_c^\infty(\Omega)$ be such that $\psi \equiv 1$ on a neighborhood of C. Then by the solution of Exercise 4 of Section 8.B we have that $\tilde{P}w = \tilde{P}(\psi w)$ on U for all $w \in \mathcal{D}'(\Omega)$. Hence we have that

$$\left(\tilde{P}u\right)\Big|_U = \left(\tilde{P}(\psi u)\right)\Big|_U = \left(p(D)(\phi\psi u) + R(\psi u)\right)\Big|_U.$$

By expanding $\operatorname{supp}\psi$ earlier if necessary, we can assume without loss of generality that $\psi \equiv 1$ on $\operatorname{supp}\phi$ as well and so the quantity $p(D)(\phi\psi u)$ above is simply $p(D)(\phi u)$. By (5.18) and the fact that $R \in \Psi^{-\infty}(\Omega)$ is smoothing, we see that the above quantity is smooth. Since $U \subseteq \Omega$ was

[2] P would be enough to prove that $(x_0, \xi_0) \notin WF(u)$ if we knew that it was properly supported. Unfortunately we don't know that (or at least I don't), and so that is the purpose here of taking the negligible modification $\tilde{P}$ of P to do the job.

chosen arbitrarily, this shows that indeed $\tilde{P}u \in C^\infty(\Omega)$. Now, considering that $\operatorname{char}_0 P = \operatorname{char}_0 \tilde{P}$, we have that $(x_0, \xi_0) \notin \operatorname{char}_0 \tilde{P}$. It's easy to see then that by Definition 2.36, $(x_0, \xi_0) \notin WF(u)$.

Now let's prove the other direction: suppose that $(x_0, \xi_0) \notin WF(u)$. Then, there exists a properly supported $P \in \Psi^0(\Omega)$ such that $Pu \in C^\infty(\Omega)$ and $(x_0, \xi_0) \notin \operatorname{char}_0 P$. Since $T^0 \setminus \operatorname{char}_0 P$ is open we have that there exists a neighborhood $N \subseteq \Omega$ such that $(x, \xi_0) \notin \operatorname{char}_0 P$ for all $x \in N$ and hence $(x, \xi_0) \notin WF(u)$ for all $x \in N$. Let $\phi \in C_c^\infty(\Omega)$ be such that $\operatorname{supp} \phi \subseteq N$ and $\phi(x_0) \neq 0$. Let

$$\Sigma = \{\xi \in \mathbb{R}^n \setminus \{0\} : (x, \xi) \in WF(\phi u) \text{ for some } x \in \Omega\},$$

which is of course a cone. Notice that by Theorem 8.54 in the book,

$$WF(\phi u) \subseteq WF(u) \cap \big(\operatorname{supp} \phi \times (\mathbb{R}^n \setminus \{0\})\big) \tag{5.19}$$

(to see why, consider the $\Psi^0(\Omega)$ operator of multiplication by ϕ). It's not hard to see that this implies that $\xi_0 \notin \Sigma$. Now, I claim that Σ is closed. To see why, let $\{\xi_k : k \in \mathbb{Z}_+\}$ be a sequence in Σ that converges to some $\xi \in \mathbb{R}^n \setminus \{0\}$. By definition, for each $k \in \mathbb{Z}_+$ there exists an $x_k \in \Omega$ such that $(x_k, \xi_k) \in WF(\phi u)$. By (5.19) we in fact know that each x_k must be contained in $\operatorname{supp} \phi$, and so by its compactness and passing to a subsequence if necessary, we may assume that $x_k \to x$ for some $x \in \operatorname{supp} \phi$. Then $(x_k, \xi_k) \to (x, \xi)$ and hence the closedness of $WF(\phi u)$ implies that $(x, \xi) \in WF(\phi u)$. Thus $\xi \in \Sigma$ and so indeed Σ is closed.

Similarly to the first part of this proof, we may choose a $p \in C^\infty(\mathbb{R}^n)$ such that $p(\xi)$ is homogeneous of degree 0 for large ξ, $p(\xi_0) \neq 0$, and p is zero on a neighborhood of Σ. Consider $p(D)[\phi u]$. Notice that its wavefront set is empty since by Theorem 8.54 in the book again we have that

$$WF(p(D)[\phi u]) \subseteq WF(\phi u) \cap \mu\operatorname{supp}\big(p(D)\big) = \emptyset.$$

Hence by Theorem 8.53 in the book we get that $p(D)[\phi u] \in C^\infty(\Omega)$. Let's now study its growth rate.

By the "claim" at the beginning of the proof, we have that $p(D)[\phi u] = \check{p} * [\phi u]$. Now, I claim that this function $\check{p} * [\phi v]$ agrees with a Schwartz function away from $\operatorname{supp} \phi$. To prove this, let $\sigma \in C^\infty(\mathbb{R}^n)$ be such that it vanishes in a neighborhood of 0 and $\sigma \equiv 1$ on a neighborhood of the region $|y| \geq 1$. I claim that

$$\check{p} * [\phi u] = [\sigma \check{p}] * [\phi u] \qquad \text{on } \mathbb{R}^n \setminus \Big(\operatorname{supp} \phi + \overline{B_1(0)}\Big). \tag{5.20}$$

To see this, take any test function $\psi \in C_c^\infty(\Omega)$ whose support is disjoint from the compact set $\operatorname{supp} \phi + \overline{B_1(0)}$. We have that

$$\langle \check{p} * [\phi u], \psi \rangle = \langle \check{p}(y), \langle \phi u(x), \psi(x + y) \rangle \rangle.$$

It's not hard to see that by assumption on ψ we have that $\langle \phi u(x), \psi(x+y)\rangle$ is zero on $\overline{B_1(0)}$ (as a function of y). Hence the above quantity can further be rewritten as

$$\langle \sigma\check{p}(y), \langle \phi u(x), \psi(x+y)\rangle\rangle = \langle [\sigma\check{p}] * [\phi u], \psi\rangle,$$

which of course establishes (5.20). Having this in hand, and reminding ourselves that $\check{p}$ agrees with a Schwartz function away from zero, we have that $\sigma\check{p} \in C^\infty(\mathbb{R}^n)$ and so we can write

$$[\sigma\check{p}] * [\phi u](x) = \langle \phi u(y), \sigma\check{p}(x-y)\rangle.$$

Utilizing the fact that $\sigma\check{p}$ is in Schwartz, it's not hard to show that this quantity is bounded over $x \in \mathbb{R}^n$ even when multiplied by any polynomial in x. Hence this quantity is a Schwartz function, which recall our original function of interest $p(D)[\phi u] = \check{p} * [\phi u]$ agrees with on the region in (5.20). Having already showed that $p(D)[\phi u] \in C^\infty(\Omega)$, we finally get that $\check{p} * [\phi u]$ is a Schwartz function. That means that its Fourier transform $p \cdot \widehat{\phi u}$ is Schwartz. It's not hard to see that this implies that $\widehat{\phi u}$ is rapidly decreasing in some open cone containing ξ_0 (i.e. satisfies the conclusion in the theorem). This proves the theorem.

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5.9 Change of Coordinates for Pseudodifferential Operators

In this section I work through the proof of the Theorem 8.58 in the book. Before we get to it however, we need to establish a few things. All diffeomorphisms are assumed to be C^∞.

Definition 5.21: *Suppose that $F : \Omega' \to \Omega$ is a diffeomorphism between open sets and that $T : C_c^\infty(\Omega) \to C^\infty(\Omega)$ is a map. Then the* ***pullback of T****, denoted by $T^F : C_c^\infty(\Omega') \to C^\infty(\Omega')$, is the map given by*

$$T^F u(x) = [T(u \circ F^{-1})] \circ F.$$

Notation 5.22: *Suppose that $F : \Omega' \to \Omega$ is a diffeomorphism between open sets. We let J_F denote the Jacobian matrix of F (i.e. $[J_F(x)]_{jk} = \partial F_j / \partial x_k\,(x)$).*

Lemma 5.23: *Suppose that $F : \Omega' \to \Omega$ is a diffeomorphism between open sets and that $L = \sum_{|\alpha|\le k} a_\alpha \partial^\alpha$ is a linear partial differential operator over Ω with C^∞ coefficients. Then L^F is also a linear partial differential operator over Ω' with C^∞ coefficients of the same order. Furthermore, the characteristic form of L^F is given by*

$$\chi_{L^F}(x,\xi) = \sum_{|\alpha|=k} a_\alpha \circ F(x)([J_F^\top(x)]^{-1}\xi)^\alpha,$$

where "...$^\top$" denotes the transpose of a matrix.

The above lemma follows from the discussion on pages 32 – 33 in the book, and so I omit its proof.

Theorem 5.24: *Suppose that $F : \Omega' \to \Omega$ is a diffeomorphism between open subsets of $\mathbb{R}^n$ and that $P = p(x, D) \in \Psi^m(\Omega)$ is a properly supported pseudodifferential operator over Ω. Then $P^F \in \Psi^m(\Omega')$ and is properly supported as well. Furthermore*

$$\sigma_{P^F}(x,\xi) = p(F(x), [J_F^\top(x)]^{-1}\xi) \quad (mod\, S^{m-1}(\Omega'))$$

(see Notation 5.11 for σ_{P^F}).

Proof: Allow us to point out that the theorem is already established when P is a linear partial differential operator with C^∞ coefficients by Lemma 5.23.

Ok, let's first show that $P^F : C_c^\infty(\Omega) \to C^\infty(\Omega)$ is properly supported. Let K^P denote the distribution kernels of P. For any $u, v \in C_c^\infty(\Omega')$ we have that

$$\langle P^F u, v\rangle = \int [P(u\circ F^{-1})\circ F](x)v(x)dx = \int P(u\circ F^{-1})(y)[v\circ F^{-1}](y)|\det J_{F^{-1}}(y)|dy$$
$$= \langle K^P(x,y), [|\det J_{F^{-1}}(x)|v\circ F^{-1}(x)] \otimes [u\circ F^{-1}(y)]\rangle.$$

Letting $G : \Omega' \times \Omega' \to \Omega \times \Omega$ denote the diffeomorphism $G(x,y) = (F(x), F(y))$ (where $x, y \in \Omega'$), we can further rewrite the above quantity as

$$\langle |\det J_F(y)| G^* K^P(x,y), v(x) \otimes v(y)\rangle$$

where G^*K^P denotes the pullback of K^P under G. Thus the distribution kernel of P^F is given by

$$K^{P^F}(x,y) = |\det J_F(y)| G^* K^P(x,y)$$

and hence

$$\operatorname{supp} K^{P^F} \subseteq G^{-1}[\operatorname{supp} K^P].$$

It's not hard to see that G^{-1} takes proper subsets to proper subsets. So, we have that P^F is indeed properly supported.

For later purposes, observe that the kernel K^{P^F} is smooth away from the diagonal $\Delta_{\Omega'} = \{(x,y) \in \Omega' \times \Omega' : x = y\}$ since by Theorem 8.8 in the book we know that K^P is smooth away from the diagonal $\Delta_\Omega = \{(x,y) \in \Omega \times \Omega : x = y\}$.

Now let's prove that $P^F \in \Psi^m(\Omega')$ and that it has the symbol stated in the theorem. Let's start with the case $m < -n$. By Lemma 8.57 in the book there exists a smooth map $\mu : N \to \mathrm{GL}_n(\mathbb{R})$ over an open neighborhood $N \subseteq \Omega' \times \Omega'$ of $\Delta_{\Omega'}$ such that $\mu(x,x) = J_F(x)$ for all $x \in \Omega'$ and

$$F(x) - F(y) = \mu(x,y)(x-y) \qquad \forall x, y \in N.$$

By Proposition 8.15 in the book there exists a $\phi \in C^\infty(\Omega' \times \Omega')$ that is identically one in a neighborhood of $\Delta_{\Omega'}$ and such that $\operatorname{supp}\phi$ is proper and contained in N. Now, we have that for any $u \in C_c^\infty(\Omega')$,

$$P^F u(x) = \iint e^{2\pi i (F(x)-z)\cdot\xi} p(F(x),\xi) u\circ F^{-1}(z) dz d\xi$$

$$= \iint e^{2\pi i\left(F(x)-F(y)\right)\cdot\xi} p(F(x),\xi)u(y)|\det J_F(y)|dyd\xi$$

$$= Qu(x) + Ru(x)$$

where

$$Qu(x) = \iint e^{2\pi i\mu(x,y)(x-y)\cdot\xi} p(F(x),\xi)u(y)|\det J_F(y)|\phi(x,y)dyd\xi,$$

$$Ru(x) = \iint e^{2\pi i\left(F(x)-F(y)\right)\cdot\xi} p(F(x),\xi)u(y)|\det J_F(y)|\big(1-\phi(x,y)\big)dyd\xi.$$

Let's first take a look at the "error term" R.

<u>Claim:</u> The distribution kernel of R is given by

$$K^R = (1-\phi)K^{P^F}.$$

<u>Proof:</u> We have that for any $u, v \in C_c^\infty(\mathbb{R}^n)$,

$$\langle Ru, v\rangle = \int \iint e^{2\pi i\left(F(x)-F(y)\right)\cdot\xi} p(F(x),\xi)u(y)|\det J_F(y)|\big(1-\phi(x,y)\big)dyd\xi\, v(x)dx$$

$$= \iint \int e^{2\pi i\left(F(x)-F(y)\right)\cdot\xi} p(F(x),\xi)|\det J_F(y)|d\xi\,\big(1-\phi(x,y)\big)v(x)u(y)dxdy$$

where the interchanging of the integrals in the last step is justified because the integrand is absolutely convergent since $m < -n$. By similar logic, the expression for $\langle P^F u, v\rangle$ is exactly the same except that it doesn't have the $(1-\phi(x,y))$ term inside of it. Thus, we get that

$$\langle Ru, v\rangle = \langle K^{P^F}, (1-\phi)[v \otimes u]\rangle.$$

From here the claim follows.

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Back to proving the theorem. Because of the above claim we see that $\operatorname{supp} K^R$ is proper. Furthermore, since K^{P^F} is smooth away from the diagonal $\Delta_{\Omega'}$ and $(1-\phi) \equiv 0$ in a neighborhood of this diagonal, we have that K^R is also smooth everywhere. In particular we get that for any compact subset $A \subseteq \Omega'$ there exists a compact subset $B \subseteq \Omega'$ such that for any $x \in A$, the support of $K(x,\ \cdot\ ) \in C_c^\infty(\Omega')$ is contained in B. Thus for any $u \in C_c^\infty(\Omega')$ we can do

$$Ru(x) = \int K^R(x,y)u(y)dy = \iint e^{2\pi i y\cdot\xi} K^R(x,y)\hat{u}(\xi)d\xi dy$$

$$= \int e^{2\pi i x\cdot\xi}\left[e^{-2\pi i x\cdot\xi}\int e^{2\pi i y\cdot\xi}K^R(x,y)dy\right]\hat{u}(\xi)d\xi = \int e^{2\pi i x\cdot\xi} r(x,\xi)\hat{u}(\xi)d\xi,$$

where

$$r(x,\xi)=e^{-2\pi i x\cdot\xi}\int e^{2\pi i y\cdot\xi}K^R(x,y)dy.$$

Using integration by parts (see the idea in the proof of Lemma 5.4) it's not hard to see that $r\in S^{-\infty}(\Omega')$. Since clearly $R=r(x,D)$, this shows that $R\in\Psi^{-\infty}(\Omega')$.

Thus, for the case $m<-n$ it suffices to show that $Q\in\Psi^m(\Omega')$ and that it has the symbol stated in the theorem. As before, since $m<-n$, we have that the integrand in the integral defining Q is absolutely convergent. Hence we can interchange the order of integration, make the substitution $\xi=\mu^{\top,-1}(x,y)\eta$ (transpose and inverse of μ), and then switch the order of integration back to get that

$$Qu(x)=\iint e^{2\pi i(x-y)\cdot\eta}p(F(x),\mu^{\top,-1}(x,y)\eta)u(y)|\det J_F(y)|\phi(x,y)|\det\mu^{\top,-1}(x,y)|dyd\eta$$

$$=\iint e^{2\pi i(x-y)\cdot\eta}a(x,\eta,y)u(y)dyd\eta$$

where

$$a(x,\eta,y)=p(F(x),\mu^{\top,-1}(x,y)\eta)|\det J_F(y)|\phi(x,y)|\det\mu^{\top,-1}(x,y)|.$$

It's not hard to see that $a\in A^m(\Omega')$ and so $Q=P_a$ with respect to Notation 2.25. By reasoning similar to what we did with R above we have that the distribution kernel of Q is given by $K^Q=\phi K^{P^F}$ and thus Q is also properly supported. Hence, by Theorem 5.7 we have that $Q\in\Psi^m(\Omega')$ and

$$\sigma_Q(x,\eta)=p(F(x),\mu^{\top,-1}(x,x)\eta)|\det J_F(x)|\phi(x,x)|\det\mu^{\top,-1}(x,x)|\quad(\mathrm{mod}\,S^{m-1}(\Omega'))$$

$$=p\big(F(x),J_F^{\top,-1}(x)\eta\big)\quad(\mathrm{mod}\,S^{m-1}(\Omega')).$$

As discussed above, this proves the case $m<-n$.

Now let's look at the case $m\geq -n$. Let $M>0$ be an integer such that $m-2M<-n$ (the reason for this condition will be clear soon). Consider the elliptic (properly supported) differential operator $\Delta^M\in\Psi^{2M}(\Omega)$ (Δ is the Laplacian). By the proof of Lemma 8.41 there exists a $\zeta\in C^\infty(\Omega\times\mathbb{R}^n)$ such that ζ is zero in a neighborhood of Δ_Ω (this condition isn't in the book) and for any compact subset $A\subseteq\Omega$ there exist constants $c,C>0$ such that for $x\in A$

1.) $\zeta(x,\xi)=1$ when $|\xi|\geq C$,

2.) $\left|\sigma_{\Delta^M}(x,\xi)\right|\geq c|\xi|^m$ when $\zeta(x,\xi)\neq 0$.

By the proof of Theorem 8.42 in the book, there exists a (properly supported) parametrix $S\in\Psi^{-2m}(\Omega)$ for Δ^M such that

$$\sigma_S(x,\xi)=\zeta(x,\xi)/\sigma_{\Delta^M}(x,\xi)\quad(\mathrm{mod}\,S^{m-1}(\Omega)).$$

where the fraction on the right-hand side is interpreted to be zero when $\zeta(x,\xi)=0$. Now, since $(S\Delta^M-I)\in\Psi^{-\infty}(\Omega)$ we have that $P=PS\Delta^M-T$ for some $T\in\Psi^{-\infty}(\Omega)$. Since $PS\in$

$\Psi^{m-2M}(\Omega)$, we can apply the already established cases of this theorem that we're proving to PS, Δ^M, and T to get that $(PS)^F \in \Psi^{m-2M}(\Omega')$, $\Delta^M \in \Psi^{2M}(\Omega')$, and $T^F \in \Psi^{-\infty}(\Omega')$ with symbols

$$\sigma_{(PS)^F}(x,\xi)$$
$$= p\left(F(x), J_F^{\top,-1}(x)\xi\right)\zeta\left(F(x), J_F^{\top,-1}(x)\xi\right)/\sigma_{\Delta^M}\left(F(x), J_F^{\top,-1}(x)\xi\right) \quad (\operatorname{mod} S^{m-2M-1}(\Omega')),$$

$$\sigma_{(\Delta^M)^F}(x,\xi) = \sigma_{\Delta^M}\left(F(x), J_F^{\top,-1}(x)\xi\right) \quad (\operatorname{mod} S^{2M-1}(\Omega')).$$

Since $P^F = (PS)^F(\Delta^M)^F - T^F$, by Corollary 8.38 in the book we get that

$$\sigma_{P^F}(x,\xi) = p\left(F(x), J_F^{\top,-1}(x)\xi\right)\zeta\left(F(x), J_F^{\top,-1}(x)\xi\right) \quad (\operatorname{mod} S^{m-1}(\Omega'))$$
$$= p\left(F(x), J_F^{\top,-1}(x)\xi\right) \quad (\operatorname{mod} S^{m-1}(\Omega')).$$

This proves the theorem.

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