Haim's Notes About

Real Analysis (2nd Ed)

by Gerald B. Folland

Table of Contents

Chapter 5 1

Net Completeness Equivalent to Sequence Completeness in First Countable Topological Vector Space (Page 167 or Problem 5.44) (9/25/2020) 1

Separately Continuous Bilinear Maps Between Banach Spaces are Continuous (Problem 5.39) (11/2/2020) 2

Equivalent Condition for Bilinear Maps between Fréchet Spaces to be Continuous (Page 166) (11/3/2020) 3

Chapter 5

Net Completeness Equivalent to Sequence Completeness in First Countable Topological Vector Space (Page 167 or Problem 5.44) (9/25/2020)

The precise statement of the result that I want to discuss is:

Theorem: *Suppose that $\mathcal{X}$ is a first countable topological vector space. Then every Cauchy net being convergent in $\mathcal{X}$ is equivalent to every Cauchy sequence being convergent.*

Proof: If every Cauchy net is convergent in $\mathcal{X}$, then obviously all Cauchy sequences converge in it since Cauchy sequences are Cauchy nets. So let's prove the other direction. Suppose that all Cauchy sequences converge in $\mathcal{X}$. Take any net $\langle x_\lambda \rangle_{\lambda \in \Lambda}$ in $\mathcal{X}$ that's Cauchy. We want to show that it converges to something. Let $U_1 \supseteq U_2 \supseteq U_3 \supseteq \cdots$ be a nested neighborhood basis of 0. Let's construct the sequence $\{x_{\gamma_k}\}_{k=1}^{\infty}$ inductively as follows. First, let $\alpha_1, \beta_1 \in \Lambda$ be such that for any $(\alpha, \beta) \gtrsim (\alpha_1, \beta_1)$ (i.e. both $\alpha \gtrsim \alpha_1$ and $\beta \gtrsim \beta_1$), $(x_\alpha - x_\beta) \in U_1$. Let $\gamma_1 \in \Lambda$ be such that $\gamma_1 \gtrsim \alpha_1, \beta_1$. Now, suppose that γ_k is defined for $k \in \{1, \dots, n-1\}$. Define γ_n as follows. As before, let $\alpha_n, \beta_n \in \Lambda$ be such that for any $(\alpha, \beta) \gtrsim (\alpha_n, \beta_n)$, $(x_\alpha - x_\beta) \in U_1$. Then let $\gamma_n \in \Lambda$ be such that $\gamma_n \gtrsim \gamma_{n-1}, \alpha_n, \beta_n$. Great! Now that we have this sequence $\{x_{\gamma_k}\}_{k=1}^{\infty}$, first notice that by construction $\gamma_1 \lesssim \gamma_2 \lesssim \gamma_3 \lesssim \cdots$. Notice also that for any integer $N > 0$ and integers $m, n \geq N$, $(x_{\gamma_m} - x_{\gamma_n}) \in U_N$ since both $\gamma_m, \gamma_n \gtrsim \gamma_N$ and $\gamma_N \gtrsim \alpha_N, \beta_N$. So the sequence $\{x_{\gamma_k}\}_{k=1}^{\infty}$ is Cauchy and hence convergent to some point $x \in \mathcal{X}$ in $\mathcal{X}$ by assumption. Now, let's prove that the original net $\langle x_\lambda \rangle_{\lambda \in \Lambda}$ also converges to x.

Consider the continuous function $f : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ given by $f(y,z) = x + y + z$. Take any open neighborhood U of x. Since $f(0,0) = x$ and f is continuous, there exists an integer $k_0 > 0$ such that $f\left[U_{k_0} \times U_{k_0}\right] \subseteq U$. Now, there exists an integer $k_1 > 0$ such that for any $k \geq k_1$, $\left(x_{\gamma_k} - x\right) \in U_{k_0}$. Let $k_2 = \max\{k_0, k_1\}$. For any $\lambda \gtrsim \gamma_{k_2}$, consider the expression:

$$x_\lambda = x + \left(x_{\gamma_{k_2}} - x\right) + \left(x_\lambda - x_{\gamma_{k_2}}\right).$$

Since $k_2 \geq k_1$, the second term on the right-hand side is in U_{k_0}: $\left(x_{\gamma_{k_2}} - x\right) \in U_{k_0}$. And since both $\lambda, \gamma_{k_2} \gtrsim \gamma_{k_0}$, we have that the third term on the right-hand side is also in U_{k_0}: $\left(x_\lambda - x_{\gamma_{k_2}}\right) \in U_{k_0}$. Thus we have that:

$$x_\lambda \in x + U_{k_0} + U_{k_0} \subseteq U,$$

and so indeed we get that $\langle x_\lambda \rangle_{\lambda \in \Lambda} \to x$. This proves what we wanted.

∎

Separately Continuous Bilinear Maps Between Banach Spaces are Continuous (Problem 5.39) (11/2/2020)

Theorem: *Suppose that $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ are Banach spaces and that $F : \mathcal{X} \times \mathcal{Y} \to \mathcal{Z}$ is a separately continuous bilinear map. Then F is continuous.*

Proof: I claim that if we prove that there exists a $C > 0$ such that

$$\|F(x,y)\| \leq C\|x\|\|y\| \qquad \forall x \in \mathcal{X} \quad \forall y \in \mathcal{Y},$$

then this will suffice. To see why, take any $(x_0, y_0) \in \mathcal{X} \times \mathcal{Y}$ and any $\varepsilon > 0 : \varepsilon < 1$. Let

$$\delta = \frac{\varepsilon/(2C)}{1 + \max\{\|x_0\|, \|y_0\|\}}.$$

Then we have that:

$$F[B_\delta(x_0) \times B_\delta(y_0)] \subseteq B_\varepsilon\big(F(x_0, y_0)\big)$$

since for any $x \in B_\delta(x_0)$ and any $y \in B_\delta(y_0)$,

$$\|F(x,y) - F(x_0,y_0)\| = \|F(x,y) - F(x_0,y) + F(x_0,y) - F(x_0,y_0)\|$$

$$= \|F(x - x_0, y) + F(x_0, y - y_0)\| \leq C\|x - x_0\|\|y\| + C\|x_0\|\|y - y_0\|$$

$$< C\delta(\|y_0\| + 1) + C\|x_0\|\delta \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

So let's prove the first inequality above. Since $y \mapsto F(x,y)$ is linear and continuous for all $x \in \mathcal{X}$, we have that for any $x \in \mathcal{X}$ there exists a constant $C_x > 0$ such that:

$$\|F(x,y)\| \leq C_x\|y\| \qquad \forall y \in \mathcal{Y}.$$

This means that

$$\sup_{y \in \mathcal{Y} : \|y\|=1} \|F(x,y)\| < \infty \qquad \forall x \in \mathcal{X}.$$

The uniform boundedness principle then says that

$$\sup_{y \in \mathcal{Y} : \|y\|=1} \|F(\,\cdot\,, y)\| = \sup_{y \in \mathcal{Y} : \|y\|=1} \sup_{x \in \mathcal{X} : \|x\|=1} \|F(x,y)\| < C < \infty$$

for some $C > 0$. For any $x \in \mathcal{X} : x \neq 0$ and any $y \in \mathcal{Y} : y \neq 0$ we then have that

$$\|F(x,y)\| = \left\| F\left(\frac{x}{\|x\|}, \frac{y}{\|y\|}\right) \right\| \|x\| \|y\| \leq C \|x\| \|y\|.$$

which is the inequality that we wanted in the beginning of the proof. If one of $x = 0$ or $y = 0$ then that inequality obviously holds as well. With this the theorem is proved.

■

Equivalent Condition for Bilinear Maps between Fréchet Spaces to be Continuous (Page 166) (11/3/2020)

Theorem: *Suppose that $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ are Fréchet space generated by the countable family of seminorms $\{p_k\}_{k=1}^{\infty}$, $\{q_k\}_{k=1}^{\infty}$, and $\{r_k\}_{k=1}^{\infty}$ respectively. Then a bilinear map $T : \mathcal{X} \times \mathcal{Y} \to \mathcal{Z}$ is continuous if and only if for any $k_0 \in \mathbb{Z}_+$ there exists a $C > 0$ and <u>finite</u> subsets $J_1, J_2 \subseteq \mathbb{Z}_+$ such that*

$$r_{k_0}\big(T(x,y)\big) \leq C \left(\sum_{k \in J_1} p_k(x) \right) \left(\sum_{s \in J_2} q_s(y) \right)$$

for all $x \in \mathcal{X}$ and $y \in \mathcal{Y}$.

Proof: This is proved exactly the same way the analog theorem is proved for linear maps of the form $T : \mathcal{X} \to \mathcal{Y}$. First let's prove the backwards implication. Take any point (x_0, y_0) and let $\langle (x_\lambda, y_\lambda) \rangle_{\lambda \in \Lambda}$ be a net that converges to (x_0, y_0) in $\mathcal{X} \times \mathcal{Y}$. Take any $k_0 \in \mathbb{Z}_+$. Let $C > 0$ and $J_1, J_2 \subseteq \mathbb{Z}_+$ be as in the statement of the theorem. Then

$$r_{k_0}\big(T(x_\lambda, y_\lambda) - T(x_0, y_0)\big) = r_{k_0}\big(T(x_\lambda - x_0, y_\lambda) + T(x_0, y_\lambda - y_0)\big)$$

$$\leq C \left(\sum_{k \in J_1} p_k(x_\lambda - x_0) \right) \left(\sum_{s \in J_2} q_s(y_\lambda) \right) + C \left(\sum_{k \in J_1} p_k(x_0) \right) \left(\sum_{s \in J_2} q_s(y_\lambda - y_0) \right)$$

$$\leq C \left[\left(\sum_{k \in J_1} p_k(x_\lambda - x_0) \right) \left(\sum_{s \in J_2} [q_s(y_\lambda - y_0) + q_s(y_0)] \right) + \left(\sum_{k \in J_1} p_k(x_0) \right) \left(\sum_{s \in J_2} q_s(y_\lambda - y_0) \right) \right],$$

which goes to zero. Hence $T(x_\lambda, y_\lambda) \to T(x_0, y_0)$ and thus T is indeed continuous.

Now let's prove the forward direction. Take any $k_0 \in \mathbb{Z}_+$. Since T is continuous at zero, we have that there exist basis neighborhoods of zero in $\mathcal{X}$ and $\mathcal{Y}$:

$$A = \{x \in \mathcal{X} : p_k(x) < \delta_k \quad \forall k \in J_1\} \qquad \text{where } J_1 \text{ is a finite subset of } \mathbb{Z}_+,$$

$$B = \{y \in \mathcal{X} : q_s(y) < \varepsilon_s \quad \forall s \in J_2\} \qquad \text{where } J_2 \text{ is a finite subset of } \mathbb{Z}_+,$$

such that $T[A \times B]$ is contained in the following open neighborhood of zero in $\mathcal{Z}$:

$$\{z \in \mathcal{Z} : r_{k_0}(z) < 1\}.$$

Letting $\delta = \min\{\delta_k : k \in J_1\}$ and $\varepsilon = \min\{\varepsilon_s : s \in J_2\}$, we have that the above implies that:

$$\begin{matrix} x \in \mathcal{X} : \sum_{k \in J_1} p_k(x) < \delta \\ y \in \mathcal{Y} : \sum_{s \in J_2} q_s(y) < \varepsilon \end{matrix} \quad \Rightarrow \quad r_{k_0}\big(T(x,y)\big) < 1$$

Let $p : \mathcal{X} \to [0,\infty)$ and $q : \mathcal{Y} \to [0,\infty)$ be the functions:

$$p(x) = \sum_{k \in J_1} p_k(x),$$

$$q(y) = \sum_{s \in J_2} q_s(y),$$

which are clearly seminorms as well. Now, take any $(x,y) \in \mathcal{X} \times \mathcal{Y}$. There are three cases that could happen:

Case $p(x) = 0$ or $p(y) = 0$: First let's suppose that $p(x) = 0$. Let $b > 0$ be such that $q(by) < \varepsilon$. Then for any $t > 0$ we have that

$$r_{k_0}\big(T(tx,by)\big) < 1$$

which can be rewritten as:

$$r_{k_0}\big(T(x,y)\big) < \frac{1}{tb}.$$

Letting $t \to \infty$ then shows that $r_{k_0}\big(T(x,y)\big) = 0$. If $p(x) \neq 0$ and $p(y) = 0$ instead, then mathematics similar to the above shows that $r_{k_0}\big(T(x,y)\big) = 0$ in this case as well.

Case $p(x) \neq 0$ and $p(y) \neq 0$: Then we have that:

$$r_{k_0}\big(T(x,y)\big) = \frac{p(x)}{\delta/2} \cdot \frac{q(y)}{\varepsilon/2} r_{k_0}\left(T\left(\frac{\delta/2}{p(x)}x, \frac{\varepsilon/2}{q(y)}y\right)\right) < \frac{p(x)}{\delta/2} \cdot \frac{q(y)}{\varepsilon/2} \cdot 1$$

$$= \frac{4}{\delta\varepsilon}\left(\sum_{k\in J_1} p_k(x)\right)\left(\sum_{s\in J_2} q_s(y)\right).$$

So on both cases we get that if we set $C = 4/(\delta\varepsilon)$ in the statement of the theorem, then we have that

$$r_{k_0}\big(T(x,y)\big) \leq C\left(\sum_{k\in J_1} p_k(x)\right)\left(\sum_{s\in J_2} q_s(y)\right).$$

Since this holds for all $x \in \mathcal{X}$ and $y \in \mathcal{Y}$, this proves the theorem.

■