Haim's Notes About

Introduction to the Theory of Distributions

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2 Notations and Conventions

Notation 2.1: For any point $x \in \mathbb{R}^n$, we will denote its Euclidean length by $|x|$:

$$|x| = \sqrt{x_1^2 + \cdots + x_n^2}.$$

Notation 2.2: For any $x \in \mathbb{R}^n$ and any $r > 0$, we let $B_r(x)$ denote the open ball of radius r centered at x with respect to the Euclidean distance:

$$B_r(x) = \{y \in \mathbb{R}^n : |y - x| < r\}.$$

Notation 2.3: Suppose that $X \subseteq Y \subseteq \mathbb{R}^n$ are open sets. For any function $\phi \in C_c^\infty(X)$, we let $\phi^Y \in C_c^\infty(Y)$ denote the smooth extension of ϕ to Y obtained by setting $\phi \equiv 0$ on $Y \setminus X$. It's trivial to see then that $\operatorname{supp}\phi = \operatorname{supp}\phi^Y$ and hence ϕ^Y is indeed compactly supported as well.

Notation 2.4: We let $\mathbb{Z}_+$ stand for the positive integers: $\mathbb{Z}_+ = \{1,2,3,\dots\}$.

Notation 2.5: For any $n \in \mathbb{Z}_+$, let $\mathcal{I}(n)$ denote the set of multi-indices of length n:

$$\mathcal{I}(n) = \{(\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n : \text{each } \alpha_k \geq 0\}.$$

Notation 2.6: Let $\alpha, \beta \in I(n)$. Then

1.) $\alpha \leq \beta$ (resp. $\alpha < \beta$) means that each $\alpha_k \leq \beta_k$ (resp. $\alpha_k < \beta_k$).

2.) $\alpha!$ denotes $\alpha_1! \cdot \dots \cdot \alpha_n!$.

3.) $|\alpha|$ denotes $\alpha_1 + \cdots + \alpha_n$.

4.) For any $x \in \mathbb{R}^n$, x^α denotes $x_1^{\alpha_1} \cdot \dots \cdot x_n^{\alpha_n}$.

5.) For any sufficiently differentiable function or distribution f, $\partial^\alpha f$ denotes $\partial^{\alpha_1} \dots \partial^{\alpha_n} f$.

Notation 2.7: Let $\Omega \subseteq \mathbb{R}^n$ be an open subset. We let the following denote the following spaces of *complex-valued* functions:

1.) $C^m(\Omega)$ denotes the space of k-times continuously differentiable functions over Ω. In particular, $C^\infty(\Omega)$ denotes the space of smooth functions.

2.) $C_c^m(\Omega)$ denotes the space of k-times continuously differentiable functions over Ω *with compact support*. Sometimes $C_c^\infty(\Omega)$ is also denoted by $\mathcal{D}(\Omega)$.

3.) We let $\mathcal{S}(\mathbb{R}^n)$ denotes the space of rapidly decreasing functions:

$$\mathcal{S}(\mathbb{R}^n) = \left\{\phi \in C^\infty(\mathbb{R}^n) : \left|x^\alpha \partial^\beta \phi(x)\right| < \infty \quad \forall \alpha, \beta \in \mathcal{I}(n)\right\}.$$

This space is called the **Schwartz space**.

Notation 2.8: Let $\Omega \subseteq \mathbb{R}^n$ be an open subset. We let the following denote the following space of distributions:

1.) $\mathcal{D}'(\Omega)$ denotes the space of distributions over Ω.

2.) $\mathcal{E}'(\Omega)$ denotes the space of distributions over Ω with compact support.

3.) $\mathcal{S}'(\mathbb{R}^n)$ denotes the space of tempered distributions over $\mathbb{R}^n$.

Definition 2.9: A function $f \in \mathbb{R}^n \to \mathbb{C}$ is said to be of **polynomial growth** if there exist $C, M \geq 0$ such that

$$|f(x)| \leq C(1 + |x|)^M \qquad \forall x \in \mathbb{R}^n.$$

Notation: If f is a function or distribution over $\mathbb{R}^n$ and $h \in \mathbb{R}^n$, then $\tau_h f$ stands for f shifted in the direction h:

$$\tau_h f = f(x-h)$$

(in the case f is a distribution, this means that $\tau_h f$ is the distribution $\phi \mapsto \langle f, \phi(x+h)\rangle$).

Notation 2.10: For points $\xi \in \mathbb{R}^n$, we let $\langle \xi \rangle$ denote the quantity

$$\langle \xi \rangle = (1+|\xi|^2)^{1/2}.$$

Definition 2.11: Suppose that $u \in \mathcal{E}'(\Omega)$ is a compactly supported distribution where Ω is an open subset of $\mathbb{R}^n$. We say that $\xi_0 \in \mathbb{R}^n \setminus \{0\}$ is *not* in the **frequency set** "$\Sigma(u)$" of u if there exists an open conic neighborhood $\Gamma \subseteq \mathbb{R}^n \setminus \{0\}$ of ξ_0 such that for all $N \in \mathbb{R}$ there exists a $C_N > 0$ such that

$$|\hat{u}(\xi)| \leq C_N \langle \xi \rangle^{-N} \qquad \forall \xi \in \Gamma.$$

Definition 2.12: Suppose that $u \in \mathcal{D}'(\Omega)$ is a distribution where Ω is an open subset of $\mathbb{R}^n$. We say that $(x_0, \xi_0) \in \Omega \times (\mathbb{R}^n \setminus \{0\})$ is not in the **wavefront set** "$WF(u)$" if there exists a $\phi \in C_c^\infty(\Omega)$ such that $\phi(x_0) \neq 0$ and $\xi \notin \Sigma(\phi u)$.

It's an easy exercise to show that one can add the requirement that ϕ must also be identically one in a neighborhood of x_0 without changing the definition.

Definition 2.13: Suppose that $K \in \mathcal{D}'(X \times Y)$ is a distribution, where $X \subseteq \mathbb{R}^m$ and $Y \subseteq \mathbb{R}^n$ are open subsets, such that if $\pi : X \times Y \to X$ denotes the projection map $(x, y) \mapsto x$, then the restriction $\pi : \operatorname{supp} K \to X$ is a proper map. Then we define **the pushforward of K by π**, denoted by $\pi_* K \in \mathcal{D}'(X)$, to be the following distribution. Take any $\phi \in C_C^\infty(X)$. Let $L \subseteq Y$ be a compact subset such that

$$\operatorname{supp} K \cap (\operatorname{supp} \phi \times Y) \subseteq \operatorname{supp} \phi \times L$$

(note that the left-hand side is the preimage of $\operatorname{supp} \phi$ under the map $\pi : \operatorname{supp} K \to X$), which exists because of the mentioned "proper" map. Let $\psi \in C_c^\infty(Y)$ be such that $\psi \equiv 1$ in a neighborhood of L. Then we define

$$\langle \pi_* K, \phi \rangle = \langle K, \phi \otimes \psi \rangle.$$

It's an easy exercise to show that our defined value of $\pi_* K$ at ϕ here does not depend on the choice of L and ψ. Furthermore, it's now hard to show that $\pi_* K$ is indeed a distribution (i.e. it satisfies the required continuity assumptions). Moreover, it's easy to show that the map π_* is linear.

3 Chapter 1

3.1 Principal Value Distribution x^{-1} (Problem 1.3) (9/25/2020)

The principle value distribution x^{-1} in $\mathcal{D}'(\mathbb{R})$ is defined by:

$$\langle x^{-1}, \phi \rangle = \text{p.v.} \int \frac{\phi(x)}{x} dx = \lim_{\varepsilon \to 0^+} \left(\int_{-\infty}^{-\varepsilon} \frac{\phi(x)}{x} dx + \int_{\varepsilon}^{\infty} \frac{\phi(x)}{x} dx \right).$$

To see that this is indeed a distribution, take any compact subset $K \subseteq \mathbb{R}$. Let $b > 0$ be such that $K \subseteq [-b, b]$. For any $\phi \in C_c^\infty(K)$, we have that:

$$\lim_{\varepsilon \to 0^+} \left(\int_{-\infty}^{-\varepsilon} \frac{\phi(x)}{x} dx + \int_{\varepsilon}^{\infty} \frac{\phi(x)}{x} dx \right) = \lim_{\varepsilon \to 0^+} \left(\int_{-b}^{-\varepsilon} \frac{\phi(x)}{x} dx + \int_{\varepsilon}^{b} \frac{\phi(x)}{x} dx \right)$$

$$= \lim_{\varepsilon \to 0^+} \left(\int_{\varepsilon}^{b} \frac{\phi(x) - \phi(-x)}{x} dx \right).$$

Using the fact that:

$$\phi(x) = \phi(0) + \int_0^x \phi'(s) ds = \phi(0) + x \int_0^1 \phi'(xt) dt$$

(the rightmost expression is in fact just the 1st order Taylor expression for ϕ based at 0), we can rewrite the previous expression as:

$$= \lim_{\varepsilon \to 0^+} \left(\int_{\varepsilon}^{b} \frac{\left(\phi(0) + x \int_0^1 \phi'(xt) dt\right) - \left(\phi(0) - x \int_0^1 \phi'(-xt) dt\right)}{x} dx \right)$$

$$\lim_{\varepsilon \to 0^+} \left(\int_{\varepsilon}^{b} \frac{x \int_0^1 \left(\phi'(xt) + \phi'(-xt)\right) dt}{x} dx \right) = \lim_{\varepsilon \to 0^+} \left(\int_{\varepsilon}^{b} \int_0^1 \left(\phi'(xt) + \phi'(-xt)\right) dt\, dx \right)$$

$$= \int_0^b \int_0^1 \left(\phi'(xt) + \phi'(-xt)\right) dt\, dx.$$

Where in the last equality I've used the Dominated Convergence Theorem with the observation that the integrand of the outside integral is bounded by $2 \sup \phi' < \infty$ and that its integration domain is bounded (it's overkill to cite DCT here though). Thus, for any $\phi \in C_c^\infty(\mathbb{R})$ we have the estimate:

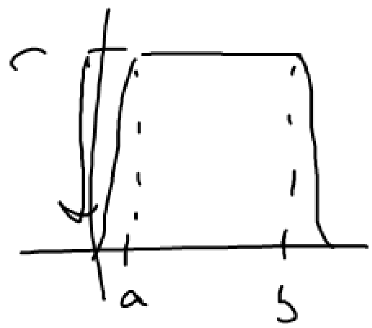
$$|\langle x^{-1}, \phi \rangle| \leq 2b \sup \phi'$$

and so x^{-1} is indeed a distribution.

Now, what is this distribution's order? It turns out to be one. To prove this, first let's observe that because of the above inequality we know that its order is less than or equal to one. So if we prove that it's not equal to zero, then we'll be done. To do this, for any positive numbers $a, b, c \in \mathbb{R} : a, b, c > 0$ let $\psi_{a,b,c} : \mathbb{R} \to \mathbb{R}$ be a compactly supported smooth bump function that satisfies:

1. $\operatorname{supp} \psi \subseteq [0, \infty)$,
2. $\psi \equiv c$ on the interval $[a, b]$,
3. $0 \leq \psi \leq c$ everywhere.

Jack Lee's Smooth Manifolds book shows how to construct such a smooth bump function. A typical example looks like:



Notice that with these functions:

$$\langle x^{-1}, \psi_{a,b,c} \rangle = \lim_{\varepsilon \to 0^+} \int_{\varepsilon}^{\infty} \frac{\psi_{a,b,c}(x)}{x} dx \geq \int_a^b \frac{c}{x} dx = c \ln\left(\frac{b}{a}\right).$$

Now, consider the sequence of functions $\{\psi_{a_k, b_k, c_k}\}_{k=1}^{\infty}$ with $a_k = 1/k$, $b_k = 1$, and $c_k = e^{-k^2}$. Then:

$$\langle x^{-1}, \psi_{a_k, b_k, c_k} \rangle = \frac{1}{k} \ln\left(\frac{1}{e^{-k^2}}\right) = k \to \infty \qquad \text{as} \quad k \to \infty.$$

But if x^{-1} was an order 0 distribution, then the distribution "seminorm estimate" would tell us that $\langle x^{-1}, \psi_{a_k, b_k, c_k} \rangle \to 0$ as $k \to \infty$ since:

$$\sup\left|\psi_{a_k, b_k, c_k}\right| = \frac{1}{k} \to 0 \qquad \text{as} \quad k \to \infty.$$

So x^{-1} must indeed be an order 1 distribution.

3.2 A Distribution $u \in \mathcal{D}'(\mathbb{R})$ such that $u = 1/x$ on $(0, \infty)$ and $u = 0$ on $(-\infty, 0)$ (Problem 1.4) (9/25/2020)

An example of a distribution $u \in \mathcal{D}'(\mathbb{R})$ such that $u = 1/x$ on $(0, \infty)$ and $u = 0$ on $(-\infty, 0)$ is:

$$\langle u, \phi \rangle = \lim_{\varepsilon \to 0^+} \int_{\varepsilon}^{\infty} \frac{\phi(x) - \phi(0)}{x} dx.$$

It obviously has the desired properties and it's obvious that the limit here exists (since the integrand tends to $\phi'(0)$ as $x \to 0^+$). To see that it's a distribution, take any compact $K \subseteq \mathbb{R}$ and any $b > 0$ such that $K \subseteq [-b, b]$. Then taking any $\phi \in C_c^\infty(K)$ and using the equation $\phi(x) = \phi(0) + x\int_0^1 \phi'(tx)dt$, we can rewrite the above quantity as:

$$\lim_{\varepsilon\to 0^+}\int_\varepsilon^b \frac{\phi(x)-\phi(0)}{x}\,dx = \lim_{\varepsilon\to 0^+}\int_\varepsilon^b\int_0^1 \phi'(tx)\,dx = \int_0^b\int_0^1 \phi'(tx)\,dx,$$

and so we get the distribution "seminorm estimate" $|\langle u, \phi\rangle| \le b\sup|\phi'|$. It's interesting to note that by the mathematics in the section discussing the principle value distribution x^{-1}, it's easy to see that this distribution has order 1. Gunther Uhlmann also observed that if we want to solve the same problem but instead require that $u = 1/x^n$ on $(0, \infty)$, then we can use:

$$\langle u, \phi\rangle = \lim_{\varepsilon\to 0^+}\int_\varepsilon^\infty \frac{\phi(x) - \sum_{k=0}^{n-1}\left(\phi^{(k)}(0)/k!\right)x^k}{x}\,dx.$$

The inner sum is of course just the $(n-1)^{\text{th}}$ Taylor polynomial of ϕ based at 0. I wonder what the order of this distribution is?

3.3 An Interesting Example of a Non-Extendible Distribution (Problem 1.5) [2/26/2021].

Consider the linear form $u : C_c^\infty(0, \infty) \to \mathbb{C}$ given by

$$\langle u, \phi\rangle = \sum_{k=1}^{\infty} \partial^k\phi(1/k).$$

The claim is that this is a distribution and that it cannot be extended to all of $\mathbb{R}$ (i.e. there does not exist a $v \in \mathcal{D}'(\mathbb{R})$ such that the restriction of v to $(0, \infty)$ is u). First let's show that it's a distribution. Take any compact subset $K \subseteq (0, \infty)$ of $(0, \infty)$. Since K is compact we have that the sequence $1/k$ for $k \in \mathbb{Z}_+$ escapes K eventually. More precisely this means that there exists an $m \in \mathbb{Z}_+$ such that for any integer $k > m$, $1/k \notin K$. Thus over K we have that u is given by the finite sum

$$\langle u, \phi\rangle = \sum_{k=1}^{m} \partial^k\phi(1/k) \qquad \forall \phi \in C_c^\infty(K).$$

Thus u satisfies the following distribution "semi-norm" estimate over K:

$$|\langle u, \phi\rangle| \le \sum_{k=1}^{m} \sup|\partial^k\phi| \qquad \forall \phi \in C_c^\infty(K).$$

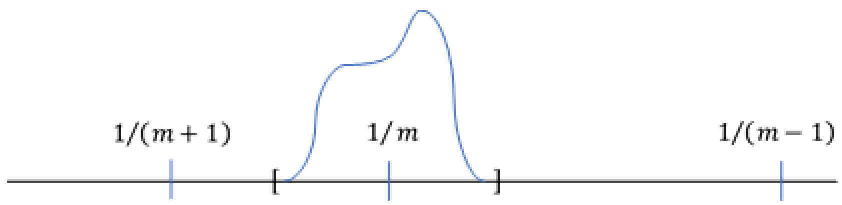
So u is indeed a distribution. Next let's prove that u cannot be extended to all of $\mathbb{R}$. Let's prove this by contradiction. Suppose not. Then there exists a distribution $v \in \mathcal{D}'(\mathbb{R})$ such that $v|_{(0,\infty)} = u$. Take the compact subset $K = [-1, 1]$ of $\mathbb{R}$ and let $C, N > 0$ be such that

$$|\langle v, \phi\rangle| \leq C \sum_{k=0}^{N} \sup|\partial^k \phi| \qquad \forall \phi \in C_c^\infty(K).$$

Let $\psi \in C_c^\infty(\mathbb{R})$ be a smooth bump function that's identically one on a neighborhood of zero and whose support is contained in $K = [-1, 1]$. For any $m \in \mathbb{Z}_+ : m \geq 2$, let $\phi_m \in C_c^\infty(\mathbb{R})$ be the test function

$$\phi_m(x) = \psi(2m(m+1)x) \cdot x^m.$$

(the condition $m \geq 2$ is for convenience so that as explained below $\operatorname{supp} \phi_m \subseteq K$). Graphically speaking, this test function is equal to x^m in a neighborhood of $1/m$ and whose compact support is contained in an interval that's situated between $1/(m+1)$ and $1/(m-1)$ (not including these two points):



For those interested for the more precise statement, the support of ϕ_m is contained in the closed interval centered at $1/m$ with radius half the distance from $1/(m+1)$ and $1/m$. On one hand we have that (in the last inequality here I bound x by $1/(m-1)$ since the $\phi_m \equiv 0$ past $x > 1/(m-1)$)

$$|\langle v, \phi_m\rangle| \leq C \sum_{k=0}^{N} \sup|\partial^k \phi_m| \leq C \sum_{k=0}^{N} \sum_{j=0}^{k} \binom{k}{j} \sup\left|\partial^j[\psi(2m(m+1)x)]\right| \cdot \sup\left|\partial^{k-j}(x^m)\right|$$

$$\leq C \sum_{k=0}^{N} \sum_{j=0}^{k} \binom{k}{j} \sup\left|\partial^j \psi\right| \left(2m(m+1)\right)^j m \cdot \ldots \cdot (m-j+1) \left(\frac{1}{m-1}\right)^{m-j}.$$

By choosing a big enough constant $D > 0$, we can estimate this bound further by

$$|\langle v, \phi_m\rangle| \leq D \frac{m^N (m+1)^N m^{N-1}}{(m-1)^{m-N}}.$$

On the other hand, since $\operatorname{supp} \phi_m \subseteq (0, \infty)$ we have that:

$$\langle v, \phi_m\rangle = \langle u, \phi_m|_{(0,\infty)}\rangle = \partial^m \phi_m(1/m) = m!$$

But we then have a contradiction since the previous inequality implies that $\langle v, \phi_m\rangle \to 0$ as $m \to \infty$ while the above equation implies that $\langle v, \phi_m\rangle \to \infty$ as $m \to \infty$. So indeed no such extension of u can exist.

4 Chapter 5

4.1 Convolution Equations on Forward Cones (Problem 5.5) (1/2/2021)

Note: Here I will write the components of my points/vectors as superscripts. For example, a point $x \in \mathbb{R}^n$ will be explicitly written out as $x = (x^1, \dots, x^n)$.

Here I discuss the following result which appears as a problem in the book.

Theorem: *Let $n \geq 2$ be an integer and for any point $x \in \mathbb{R}^n$ let $\tilde{x}$ denote the point obtained by projecting it down to its first $n-1$ components: $\tilde{x} = (x^1, \dots, x^{n-1})$. In addition, let Γ be the forward cone:*

$$\Gamma = \{x \in \mathbb{R}^n : x^n \geq c|\tilde{x}|\}$$

where $c > 0$ is some fixed positive constant. Let $\mathcal{D}'_\Gamma(\mathbb{R}^n)$ denote the set of all distributions over $\mathbb{R}^n$ whose support is contained in Γ:

$$\mathcal{D}'_\Gamma(\mathbb{R}^n) = \{u \in \mathcal{D}'(\mathbb{R}^n) : supp\, u \subseteq \Gamma\}.$$

Then the following are true:

a) *If $u_1, \dots, u_m \in \mathcal{D}'_\Gamma(\mathbb{R}^n)$, then the convolution $u_1 * \dots * u_m$ is well defined.*

b) *If $u_1, \dots, u_m \in \mathcal{D}'_\Gamma(\mathbb{R}^n)$ and $v \in \mathcal{D}'(\mathbb{R}^n)$ is such that $supp\, v \subseteq \{x^n \geq a\}$ for some fixed $a \in \mathbb{R}$, then the convolution $u_1 * \dots * u_m * v$ is also well defined.*

c) *Suppose that $k \in \mathcal{D}'_\Gamma(\mathbb{R}^n)$ is a convolution operator that has a fundamental solution $E \in \mathcal{D}'_\Gamma(\mathbb{R}^n)$. Then for any $v \in \mathcal{D}'(\mathbb{R}^n)$ such that $supp\, v \subseteq \{x^n \geq a\}$, there exists a unique solution u to the equation $k * u = v$ such that $supp\, u \subseteq \{x^n \geq a\}$ as well. Furthermore, for any such solution u, $supp(u) \subseteq supp\, E + supp\, v$.*

Proof: Let's start with (a). Take any $u_1, \dots, u_m \in \mathcal{D}'_\Gamma(\mathbb{R}^n)$. We need to show that the addition function is proper over $\operatorname{supp} u_1 \times \dots \times \operatorname{supp} u_m$. We will do this by using the criterion for such properness described in Definition 5.3.1 in the book. Take any $\delta > 0$ and suppose that $x_1, \dots, x_m \in \mathbb{R}^n$ are such that each $x_i \in \operatorname{supp} u_i$ and $|x_1 + \dots + x_m| \leq \delta$. Since each $\operatorname{supp} u_i \subseteq \Gamma$, we have that each $x_i^n \geq 0$ and so the previous inequality implies that each $x_i^n \leq \delta$. By definition of Γ we then get that each $|\tilde{x}_i| \leq \delta/c$ and so each:

$$|x_i| \leq \sqrt{(\delta/c)^2 + \delta^2} = \delta\sqrt{(1/c)^2 + 1}.$$

Setting $\delta' = \delta\sqrt{(1/c)^2 + 1}$ in Definition 5.3.1 then proves the desired properness. This proves (a).

Onwards to (b)! This is proved similarly: take any $u_1, \dots, u_m \in \mathcal{D}'_\Gamma(\mathbb{R}^n)$ and any $v \in \mathcal{D}'(\mathbb{R}^n)$ such that $\operatorname{supp} v \subseteq \{x^n \geq a\}$ for some fixed $a \in \mathbb{R}$. Take any $\delta > 0$ and suppose that $x_i \in \operatorname{supp} u_i$ and $y \in \operatorname{supp} v$ are such that $|x_1 + \dots + x_m + y| \leq \delta$. The last inequality implies that both $|\tilde{x}_1 + \dots + \tilde{x}_m + \tilde{y}| \leq \delta$ and $|x_1^n + \dots + x_m^n + y^n| \leq \delta$. The latter coupled with the facts that each $x_i^n \geq 0$ and $y^n \geq a$ gives:

$$|x_i^n| \leq x_1^n + \cdots + x_m^n + a - a \leq x_1^n + \cdots + x_m^n + y^n - a \leq \delta - a,$$

$$|y^n| \leq |x_1^n + \cdots + x_m^n + y^n| + |x_1^n + \cdots + x_m^n| \leq \delta + m(\delta - a).$$

By the definition of Γ we have that each $|\tilde{x}_i| \leq (\delta - a)/c$. The inequality $|\tilde{x}_1 + \cdots + \tilde{x}_m + \tilde{y}| \leq \delta$ then gives us that:

$$|\tilde{y}| \leq |\tilde{x}_1 + \cdots + \tilde{x}_m + \tilde{y}| + |\tilde{x}_1 + \cdots + \tilde{x}_m| \leq \delta + m(\delta - a)/c.$$

In total we get the results:

$$|x_i| \leq \sqrt{[(\delta - a)/c]^2 + (\delta - a)^2},$$

$$|y| \leq \sqrt{[\delta + m(\delta - a)/c]^2 + [\delta + m(\delta - a)]^2}.$$

Setting $\delta' > 0$ in Definition 5.3.1 to be any constant bigger than the two constants on the right-hand sides above then proves (b).

Finally we come to (c). Take any $v \in \mathcal{D}'(\mathbb{R}^n)$ such that $\operatorname{supp} v \subseteq \{x^n \geq a\}$ for some fixed $a \in \mathbb{R}$. Obviously $u = E * v$ is a desired solution since by Theorem 5.3.2 (iii) in the book,

$$\operatorname{supp} u \subseteq \operatorname{supp} E + \operatorname{supp} v \subseteq \{x^n \geq a\}$$

and

$$k * u = k * E * v = \delta * v = v$$

(all of these convolutions make sense by part (b)). To prove that this is the unique desired solution, suppose that $\tilde{u}$ is another such solution. Convoluting both sides of the equation $k * \tilde{u} = v$ by E from the left gives:

$$E * k * \tilde{u} = E * v.$$

The left-hand side here is equal to:

$$E * k * \tilde{u} = k * E * \tilde{u} = \delta * \tilde{u} = \tilde{u}.$$

Plugging this into the previous equation recovers our previous solution: $\tilde{u} = E * v$. So indeed the desired solution to our equation is unique.

■

5 Chapter 6

5.1 Schwartz Kernel Theorem [Maintenance planned] (11/4/2020)

Note: I plan to do much needed maintenance on this entry. In particular, I want to make it a lot more concise.

Note: This entry is hard.

Here I give my version of a proof to the Schwartz Kernel Theorem. It's exactly the same as the proof in this book except that I fill in all of the details. Since this proof involves a lot of steps, I think the best approach for any reader would be to first read Friedlander's shorter proof in order to get the main idea, and then read this proof if they want to see the details filled in.

Before we prove the theorem, let's first observe a lemma:

Lemma: *Suppose that $f \in C^\infty(\mathbb{R}^n)$ is a smooth function such that it and all of its partials are T-periodic where $T > 0$. Then its Fourier series converges uniformly to f:*

$$f(x) = \sum_{g \in \mathbb{Z}} \hat{f}_g \, e^{2\pi i (x \cdot g)/T}$$

where each

$$\hat{f}_g = \frac{1}{T^n} \int_{R_T} f(s) e^{-2\pi i (s \cdot g)/T} ds,$$

where R_T is any T-periodic box (for example: $R_T = [0, T]^n$). Furthermore, for any $\alpha \in \mathcal{I}(n)$ (see "notations and conventions"), the ∂^α partial of the Fourier series' partial sums converge uniformly to $\partial^\alpha f$.

Proof: I'd say that this is a rather standard fact from the theory of Fourier series.

■

Now for the Schwartz Kernel Theorem:

Schwartz Kernel Theorem: *Suppose that $X \subseteq \mathbb{R}^m$ and $Y \subseteq \mathbb{R}^n$ are open subsets. A linear map $\mu : C_c^\infty(Y) \to \mathcal{D}'(X)$ is sequentially continuous if and only if it's generated by a Schwartz Kernel $k \in \mathcal{D}'(X \times Y)$:*

$$\langle \mu\psi, \phi \rangle = \langle k, \phi \otimes \psi \rangle.$$

Furthermore, k here is uniquely determined by μ.

Proof: We already proved in the book that if such a map μ is generated by a Schwartz kernel $k \in \mathcal{D}'(X \times Y)$, then it's sequentially continuous. So let's prove the forward implication. The fact that k is uniquely determined by μ is obvious since the above equation determines what k is equal to on a dense subset of $C_c^\infty(X \times Y)$ (dense with respect to convergence in $C_c^\infty(X \times Y)$ of course). So let's prove the existence of such a $k \in \mathcal{D}'(X \times Y)$.

Let $\{K_j\}_{j=1}^\infty$ and $\{Q_j\}_{j=1}^\infty$ be compact exhaustions of X and Y respectively (we will need that fact that each $K_j \subseteq K_{j+1}^{\text{int}}$ and $Q_j \subseteq Q_{j+1}^{\text{int}}$ later). Consider the bilinear form $B : C_c^\infty(X) \times C_c^\infty(Y) \to \mathbb{C}$ given by:

$$B(\phi, \psi) = \langle \mu\psi, \phi \rangle.$$

Now, fix any $j_0 \in \mathbb{Z}_+$. Take any $\psi \in C_c^\infty(Q_{j_0})$. Since $\mu\psi \in \mathcal{D}'(X)$, we have that there exist $C_\psi, M_\psi > 0$ such that

$$|B(\phi,\psi)| = |\langle \mu\psi, \phi\rangle| \le C_\psi \sum_{|\alpha|\le M_\psi} \sup|\partial^\alpha \phi| \qquad \forall \phi \in C_c^\infty(K_{j_0}).$$

Now take any $\phi \in C_c^\infty(K_{j_0})$. The map $\psi \mapsto \mu\psi$ being sequentially continuous implies that the map $\psi \mapsto \langle \mu\psi, \phi\rangle$ is also sequentially continuous. Thus the latter map is a distribution in $D'(Y)$ and so there exist $C_\phi, N_\phi > 0$ such that:

$$|B(\phi,\psi)| = |\langle \mu\psi, \phi\rangle| \le C_\phi \sum_{|\beta|\le N_\phi} \sup\left|\partial^\beta \psi\right| \qquad \forall \psi \in C_c^\infty(Q_{j_0}).$$

These two inequalities show that the restriction $B_{j_0} : C_c^\infty(K_{j_0}) \times C_c^\infty(Q_{j_0}) \to \mathbb{C}$ of B is a separately continuous bilinear form over the Fréchet spaces $C_c^\infty(K_{j_0})$ and $C_c^\infty(Q_{j_0})$. Now, noting that Fréchet spaces are also Banach spaces we have by a quick corollary of the uniform boundedness principle that B_{j_0} is continuous with respect to the product topology. I prove this corollary in my electronic diary about Folland's "Real Analysis" book. In the that same electronic diary, I also prove an equivalent condition for a bilinear map between Fréchet spaces being continuous which in our case gives that there exist $C_{j_0}, N_{j_0} > 0$ such that:

$$|B(\phi,\psi)| = \left|B_{j_0}(\phi,\psi)\right| \le C_{j_0}\left(\sum_{|\alpha|\le N_{j_0}} \sup|\partial^\alpha \phi|\right)\left(\sum_{|\beta|\le N_{j_0}} \sup\left|\partial^\beta \psi\right|\right)$$

$$\forall \phi \in C_c^\infty(K_{j_0}) \ \text{ and } \ \forall \psi \in C_c^\infty(Q_{j_0}).$$

Ok, with this in hand we are now ready to begin constructing the $k \in \mathcal{D}'(X \times Y)$ that we want. We will do this by defining continuous linear forms $k_j : C_c^\infty(K_j \times Q_j) \to \mathbb{C}$ for $j \in \mathbb{Z}_+$ and then use their values to define k. Fix any $j_0 \in \mathbb{Z}_+$. Let $\rho \in C_c^\infty(K_{j_0+1})$ and $\sigma \in C_c^\infty(Q_{j_0+1})$ be such that $\rho \equiv 1$ and $\sigma \equiv 1$ on neighborhoods of K_{j_0} and Q_{j_0} respectively. Let $b > 0$ be such that $K_{j_0+1} \times Q_{j_0+1} \subseteq [-b,b]^{m+n}$. Now, take any $\chi \in C_c^\infty(K_{j_0} \times Q_{j_0})$. Define the value of k_{j_0} at χ to be:

$$\langle k_{j_0}, \chi\rangle = \sum_{(g,h)\in\mathbb{Z}^m\times\mathbb{Z}^n} B\left(\hat{\chi}_{(g,h)}\rho(x)e^{2\pi i(x\cdot g)/(2b)}, \sigma(y)e^{2\pi i(y\cdot h)/(2b)}\right)$$

where $\hat{\chi}_{(g,h)}$ are the Fourier coefficients of χ:

$$\hat{\chi}_{(g,h)} = \frac{1}{(2b)^{m+n}} \int_{[-b,b]^{m+n}} \chi(x,y)e^{-2\pi i(x\cdot g + y\cdot h)/(2b)} dxdy.$$

Technically I should be writing $\chi^{\mathbb{R}^m\times\mathbb{R}^n}$ (see notations and conventions) rather than χ inside the above integral since χ is not necessarily defined over $[-b,b]$. Ok, in order for the previous expression to make sense let's prove that the sum on the right-hand side converges uniformly. Before we do that though, I'd like to point out that once we prove that the above sum makes sense and is finite, then it will be clear that k_{j_0} is a linear form since the equation for the Fourier coefficient is linear in χ and B is linear in its first argument.

Ok, let's prove that that sum is absolutely convergent. First let's create a bound to estimate how large the above Fourier coefficients of χ can get. Fix any index $(g,h) \in \mathbb{Z}^m \times \mathbb{Z}^n$. First a piece of notation: for an any multi-index $\alpha \in \mathcal{I}(r)$, let $\mathcal{H}(\alpha)$ be the multi-index whose s^{th} component is equal to 1 if $\alpha_s > 0$ and is equal to 0 if $\alpha_s = 0$. Let $N = N_{j_0+1}$. Notice that since

$$\operatorname{supp}\chi \subseteq K_{j_0} \times Q_{j_0} \subseteq K_{j_0+1}^{\text{int}} \times Q_{j_0+1}^{\text{int}} \subseteq [-b,b]^{m+n},$$

we have that χ and all of its partials are zero on the boundary of the box $[-b,b]^{m+n}$. So for any $(g,h) \in \mathbb{Z}^m \times \mathbb{Z}^n$ we have by many integrations by parts that

$$\hat{\chi}_{(g,h)} = \frac{1}{(2b)^{m+n}}\left(\frac{2b}{-2\pi i}\right)^{(N+2)|\mathcal{H}(g,h)|} \frac{1}{\prod_{r=1,g_r\neq 0}^m g_r^{N+2}\prod_{s=1,h_s\neq 0}^n h_s^{N+2}}$$

$$\cdot \int_{[-b,b]^{m+n}} \partial^{(N+2)\mathcal{H}(g,h)}[\chi(x,y)]e^{-2\pi i(x\cdot g+y\cdot h)/(2b)}dxdy.$$

Let $E_1 > 0$ be a constant such that:

$$\left|\frac{1}{(2b)^{m+n}}\left(\frac{2b}{-2\pi i}\right)^{(N+2)|\mathcal{H}(g,h)|}\right| \leq E_1.$$

Furthermore, notice that we can remove the unpleasant $g_r \neq 0$ and $h_s \neq 0$ indexing rules above by writing the estimate (here I use the fact that $1/r \leq 2/(1+r)$ if $r > 0$ is an integer):

$$\left|\frac{1}{\prod_{r=1,g_r\neq 0}^m g_r^{N+2}\prod_{s=1,h_s\neq 0}^n h_s^{N+2}}\right| \leq \frac{2^{m+n}}{\prod_{r=1}^m(1+|g_r|^{N+2})\prod_{s=1}^n(1+|h_s|^{N+2})}$$

Setting $E_2 = 2^{m+n}E_1$, we get the following estimate on our Fourier coefficient:

$$\left|\hat{\chi}_{(g,h)}\right| \leq E_2 \frac{1}{\prod_{r=1}^m(1+|g_r|^{N+2})\prod_{s=1}^n(1+|h_s|^{N+2})} \operatorname{Vol}([-b,b]^{m+n}) \sup\left|\partial^{(N+2)\mathcal{H}(g,h)}\chi\right|$$

Setting $E_3 = E_2 \operatorname{Vol}([-b,b]^{m+n})$, notice that we can furthermore estimate our Fourier coefficient as:

$$\left|\hat{\chi}_{(g,h)}\right| \leq E_3\left(\sum_{|\gamma|\leq N+2}\sup|\partial^\gamma\chi|\right)\frac{1}{\prod_{r=1}^m(1+|g_r|^{N+2})\prod_{s=1}^n(1+|h_s|^{N+2})},$$

which is written in a bit more familiar terms. Great! We're going to use this estimate below.

Meanwhile, going back to our definition of $\langle k_{j_0}, \chi \rangle$, notice that we can bound each summand term on the right by:

$$\left| B\left(\hat{\chi}_{(g,h)} \rho(x) e^{2\pi i (x\cdot g)/(2b)}, \sigma(y) e^{2\pi i (x\cdot h)/(2b)}\right)\right|$$

$$\leq C_{j_0} \left(\sum_{|\alpha|\leq N} \sup \left| \partial^\alpha \left[\hat{\chi}_{(g,h)} \rho(x) e^{2\pi i (x\cdot g)/(2b)} \right] \right| \right) \left(\sum_{|\beta|\leq N} \sup \left| \partial^\beta \left[\sigma(y) e^{2\pi i (y\cdot h)/(2b)} \right] \right| \right)$$

Now apply the product rule on each ∂^α and ∂^βpartials (recall that by convention $0^0 = 1$ when raising something to a multi-index):

$$\left| B\left(\hat{\chi}_{(g,h)} \rho(x) e^{2\pi i (x\cdot g)/(2b)}, \sigma(y) e^{2\pi i (x\cdot h)/(2b)}\right)\right|$$

$$\leq C_{j_0} \left| \hat{\chi}_{(g,h)} \right| \left(\sum_{|\alpha|\leq N} \sum_{\eta \leq \alpha} \sup \left| \frac{\alpha!}{\eta!\,(\alpha-\eta)!} \left(\frac{2\pi i}{2b} \right)^{|\eta|} g^\eta e^{2\pi i (x\cdot g)/(2b)} \partial^{\alpha-\eta} \rho \right| \right)$$

$$\cdot \left(\sum_{|\beta|\leq N} \sum_{\nu \leq \beta} \sup \left| \frac{\beta!}{\nu!\,(\beta-\nu)!} \left(\frac{2\pi i}{2b} \right)^{|\nu|} h^\nu e^{2\pi i (y\cdot h)/(2b)} \partial^{\beta-\nu} \sigma \right| \right)$$

If we distribute above sum, we merely get a big linear combination of terms of the form $g^\eta h^\nu$. So for some collection of coefficients $A_{(\eta,\nu)}$,

$$\left| B\left(\hat{\chi}_{(g,h)} \rho(x) e^{2\pi i (x\cdot g)/(2b)}, \sigma(y) e^{2\pi i (x\cdot h)/(2b)}\right)\right| \leq C_{j_0} \left| \hat{\chi}_{(g,h)} \right| \sum_{\substack{|\eta|\leq N \\ |\nu|\leq N}} A_{(\eta,\nu)} g^\eta h^\nu.$$

Now, if we plug in the above estimate for $\left| \hat{\chi}_{(g,h)} \right|$ into this inequality we get that for some constant $E_4 > 0$,

$$\left| B\left(\hat{\chi}_{(g,h)} \rho(x) e^{2\pi i (x\cdot g)/(2b)}, \sigma(y) e^{2\pi i (x\cdot h)/(2b)}\right)\right|$$

$$\leq E_4 \sum_{\substack{|\eta|\leq N \\ |\nu|\leq N}} \frac{g^\eta h^\nu}{\prod_{r=1}^m (1+|g_r|^{N+2}) \prod_{s=1}^n (1+|h_s|^{N+2})} \sum_{|\gamma|\leq N+2} \sup |\partial^\gamma \chi|$$

Ok, since each $|\eta| \leq N$ in the first sum, we have that

$$\frac{g^\eta}{\prod_{r=1}^m (1+|g_r|^{N+2})} \leq \frac{g^\eta}{\prod_{r=1, g_r\neq 0}^m g_r^{N+2}} \leq \frac{1}{\prod_{r=1, g_r\neq 0}^m g_r^2} \leq \frac{2^m}{\prod_{r=1}^m (1+|g_r|^2)}.$$

By a similar calculation we have that

$$\frac{h^\nu}{\prod_{s=1}^n (1+|h_s|^{N+2})} \leq \frac{2^n}{\prod_{s=1}^n (1+|h_s|^2)}.$$

So we get that for some constant $E_5 > 0$,

$$\left|B\left(\hat{\chi}_{(g,h)}\rho(x)e^{2\pi i(x\cdot g)/(2b)}, \sigma(y)e^{2\pi i(x\cdot h)/(2b)}\right)\right|$$
$$\leq E_5 \frac{1}{\prod_{r=1}^m (1+|g_r|^2)\prod_{s=1}^n (1+|h_s|^2)} \sum_{|\gamma|\leq N+2} \sup|\partial^\gamma \chi|.$$

Tracing the above calculation again, notice that the value of the constant E_5 is independent of (g,h) (and even χ). So the above estimate holds for all $(g,h) \in \mathbb{Z}^m \times \mathbb{Z}^n$ (and all χ). Thus the above inequality shows that that the sum in the definition of $\langle k_{j_0}, \chi\rangle$ decays like g_r^2 and h_s^2 in all direction. Hence that sum is indeed absolutely convergent and thus makes sense. The above inequality shows more: it gives us the estimate:

$$\left|\langle k_{j_0}, \chi\rangle\right| \leq E_5 \left(\sum_{(g,h)\in\mathbb{Z}^m\times\mathbb{Z}^n} \frac{1}{\prod_{r=1}^m (1+|g_r|^2)\prod_{s=1}^n (1+|h_s|^2)}\right) \sum_{|\gamma|\leq N+2} \sup|\partial^\gamma \chi|.$$

Again, since the $\Sigma_{(g,h)\in\mathbb{Z}^m\times\mathbb{Z}^n}$ sum here is finite and the value of the constant E_5 is independent of χ, this shows that k_{j_0} is in fact a *continuous* linear form over $C_c^\infty\left(K_{j_0} \times Q_{j_0}\right)$.

Great, let's show one last property of k_{j_0}. Let's show that for any $\phi \in C_c^\infty\left(K_{j_0}\right)$ and any $\psi \in C_c^\infty\left(Q_{j_0}\right)$,

$$\langle k_{j_0}, \phi \otimes \psi\rangle = B(\phi, \psi) = \langle \mu\psi, \phi\rangle.$$

The Fourier coefficients of $(\phi \otimes \psi)^{\mathbb{R}^m\times\mathbb{R}^n}$ are given by:

$$(\widehat{\phi \otimes \psi})_{(g,h)} = \frac{1}{(2b)^{m+n}} \int_{[-b,b]^{m+n}} \phi(x)\psi(y)e^{-2\pi i(x\cdot g+y\cdot h)/(2b)}dxdy$$

$$= \frac{1}{(2b)^m} \int_{[-b,b]^m} \phi(x)e^{-2\pi i(x\cdot g)/(2b)}dx \cdot \frac{1}{(2b)^n} \int_{[-b,b]^n} \psi(y)e^{-2\pi i(y\cdot h)/(2b)}dy.$$

Setting $\hat{\phi}_g$ and $\hat{\psi}_h$ to be the Fourier coefficients of $\phi^{\mathbb{R}^m}$ and $\psi^{\mathbb{R}^n}$ respectively:

$$\hat{\phi}_g = \frac{1}{(2b)^m} \int_{[-b,b]^m} \phi(x)e^{-2\pi i(x\cdot g)/(2b)}dx,$$

$$\hat{\psi}_h = \frac{1}{(2b)^n} \int_{[-b,b]^n} \psi(x)e^{-2\pi i(y\cdot h)/(2b)}dy,$$

we then get the trivial tensor Fourier coefficient relation:

$$(\widehat{\phi \otimes \psi})_{(g,h)} = \hat{\phi}_g \cdot \hat{\psi}_h.$$

Using this in the definition of $\langle k_{j_0}, \phi \otimes \psi \rangle$ then gives us that

$$\langle k_{j_0}, \phi \otimes \psi \rangle = \sum_{(g,h)\in\mathbb{Z}^m\times\mathbb{Z}^n} B\big(\hat{\phi}_g \rho(x) e^{2\pi i (x\cdot g)/(2b)}, \hat{\psi}_h \sigma(y) e^{2\pi i (y\cdot h)/(2b)}\big).$$

Now, it's easy to see by the lemma stated before this theorem that the series

$$\sum_{g\in\mathbb{Z}^m} \hat{\phi}_g \rho(x) e^{2\pi i (x\cdot g)/(2b)} = \rho(x) \sum_{g\in\mathbb{Z}^m} \hat{\phi}_g e^{2\pi i (x\cdot g)/(2b)}.$$

converges to $\rho \cdot \phi$ in $C_c^\infty(\mathbb{R}^m)$. So this series converges to ϕ in $C_c^\infty\big(K_{j_0+1}\big)$ since $\rho \cdot \phi = \phi$ on K_{j_0+1} (recall that $\operatorname{supp} \phi \subseteq K_{j_0}$ and $\rho \equiv 1$ on K_{j_0}). For the same reason, we have that the series

$$\sum_{h\in\mathbb{Z}^n} \hat{\psi}_h \sigma(y) e^{2\pi i (y\cdot h)/(2b)}$$

converges to ψ in $C_c^\infty\big(Q_{j_0+1}\big)$ as well. Using the fact that $B_{j_0+1} : C_c^\infty\big(K_{j_0+1}\big) \times C_c^\infty\big(Q_{j_0+1}\big) \to \mathbb{C}$ is separately continuous we then have that

$$\langle k_{j_0}, \phi \otimes \psi \rangle = \sum_{g\in\mathbb{Z}^m} \sum_{h\in\mathbb{Z}^n} B_{j_0+1}\big(\hat{\phi}_g \rho(x) e^{2\pi i (x\cdot g)/(2b)}, \hat{\psi}_h \sigma(y) e^{2\pi i (y\cdot h)/(2b)}\big)$$

$$= \sum_{g\in\mathbb{Z}^m} B_{j_0+1}\big(\hat{\phi}_g \rho(x) e^{2\pi i (x\cdot g)/(2b)}, \psi\big) = B_{j_0}(\phi, \psi) = B(\phi, \psi) = \langle \mu\psi, \phi \rangle.$$

We are now finally ready for the last step in the proof of this theorem. Define the function $k : C_c^\infty(X \times Y) \to \mathbb{C}$ as follows. Take any test function $\chi \in C_c^\infty(X \times Y)$ and let $j_0 \in \mathbb{Z}_+$ be such that $\operatorname{supp} \chi \subseteq K_{j_0} \times Q_{j_0}$. Then set:

$$\langle k, \chi \rangle = \langle k_{j_0}, \chi \rangle.$$

Let's prove that this is well defined by showing that this is independent of the j_0 that we chose that satisfies the above property. Let $r_0 \in \mathbb{Z}_+$ be any other such index and let's suppose without loss of generality that $j_0 \le r_0$. As before, let $\rho \in C_c^\infty\big(K_{j_0+1}\big)$ and $\sigma \in C_c^\infty\big(Q_{j_0+1}\big)$ be such that $\rho \equiv 1$ and $\sigma \equiv 1$ on neighborhoods of K_{j_0} and Q_{j_0} respectively. Letting $b > 0$ be such that $K_{j_0+1} \times Q_{j_0+1} \subseteq [-b, b]^{m+n}$ we have by a similar discussion as above that the series

$$\sum_{(g,h)\in\mathbb{Z}^m\times\mathbb{Z}^n} \hat{\chi}_{(g,h)} \rho(x) e^{2\pi i (x\cdot g)/(2b)} \sigma(y) e^{2\pi i (y\cdot h)/(2b)}$$

converges to χ in $C_c^\infty\big(K_{j_0+1} \times Q_{j_0+1}\big)$ and thus in $C_c^\infty\big(K_{r_0+1} \times Q_{r_0+1}\big)$ (since $K_{j_0+1} \times Q_{j_0+1} \subseteq K_{r_0+1} \times Q_{r_0+1}$). The continuity and linearity of both k_{j_0} and k_{r_0} implies that their values at χ are given by:

$$\langle k_{j_0}, \chi \rangle = \sum_{(g,h)\in\mathbb{Z}^m\times\mathbb{Z}^n} k_{j_0}\big(\hat{\chi}_{(g,h)} \rho(x) e^{2\pi i (x\cdot g)/(2b)} \otimes \sigma(y) e^{2\pi i (y\cdot h)/(2b)}\big),$$

$$\langle k_{r_0}, \chi\rangle = \sum_{(g,h)\in\mathbb{Z}^m\times\mathbb{Z}^n} k_{r_0}\big(\hat{\chi}_{(g,h)}\rho(x)e^{2\pi i(x\cdot g)/(2b)} \otimes \sigma(y)e^{2\pi i(y\cdot h)/(2b)}\big),$$

both of which are equal to

$$\sum_{(g,h)\in\mathbb{Z}^m\times\mathbb{Z}^n} B\big(\hat{\chi}_{(g,h)}\rho(x)e^{2\pi i(x\cdot g)/(2b)}, \sigma(y)e^{2\pi i(y\cdot h)/(2b)}\big).$$

So $\langle k_{j_0}, \chi\rangle = \langle k_{r_0}, \chi\rangle$, and thus k is indeed well defined. It's easy to see from the definition of k that it's linear. To see that it's a distribution, we also have to show that it satisfies the distribution "seminorm" like inequalities. Take any compact set $R \subseteq X \times Y$. Let $j_0 \in \mathbb{Z}_+$ be such that $R \subseteq K_{j_0} \times Q_{j_0}$. Then since k_{j_0} is continuous we have that there exist $C, N > 0$ such that for any $\chi \in C_c^\infty(R) \subseteq C_c^\infty(K_{j_0} \times Q_{j_0})$,

$$|\langle k, \chi\rangle| = \left|\langle k_{j_0}, \chi\rangle\right| \le C \sum_{|\gamma|\le N} \sup|\partial^\gamma \chi|.$$

So k is indeed a distribution over $X \times Y$ (i.e. $k \in \mathcal{D}'(X \times Y)$). The last thing to show is that it satisfies the property that we desire. Take any $\phi \in C_c^\infty(X)$ and any $\psi \in C_c^\infty(Y)$. Let $j_0 \in \mathbb{Z}_+$ be such that $\operatorname{supp}\phi \times \operatorname{supp}\psi \subseteq K_{j_0} \times Q_{j_0}$. Then we have that:

$$\langle k, \phi \otimes \psi\rangle = \langle k_{j_0}, \phi \otimes \psi\rangle = \langle \mu\psi, \phi\rangle.$$

So k is the Schwartz kernel of our map μ. With this we've proven the theorem.

■

6 Chapter 8

6.1 Structure Theorem for Tempered Distributions [Theorem 8.3.1] (5/11/2021)

In this entry I work through the proof of the structure theorem for tempered distributions (Theorem 8.3.1 in the book) by putting it into my own words and filling in the details.

Theorem: *A distribution is a tempered distribution if and only if it is the derivative of a continuous function of polynomial growth.*

Proof: First suppose that a distribution $u \in \mathcal{D}'(\mathbb{R}^n)$ is the derivative of a continuous function $f \in C^0(\mathbb{R}^n)$ of polynomial growth:

$$u = \partial^\alpha f,$$

where $\alpha \in \mathcal{I}(n)$ of course. We want to show that u is a tempered distribution. Since f is of polynomial growth, there exist $C > 0$ and $M \in \mathbb{Z}_+$ such that $|f(x)| \le C(1 + |x|)^M$. Now, I claim that the linear functional $\mathcal{S}(\mathbb{R}^n) \to \mathbb{C}$ given by

$$(6.1) \qquad \phi \longmapsto (-1)^{|\alpha|} \int f \partial^\alpha \phi$$

is a well-defined continuous extension of the distribution $\partial^\alpha f : C_c^\infty(\mathbb{R}^n) \to \mathbb{C}$ to $\mathcal{S}(\mathbb{R}^n)$. If we prove this, then this will show that u is indeed a tempered distribution. Take any $\phi \in \mathcal{S}(\mathbb{R}^n)$. First let's show that the above integral even exists. We estimate (here m is the Lebesgue measure)

$$\int |f \partial^\alpha \phi| \leq C \int (1+|x|)^M |\partial^\alpha \phi| = C \left[\int_{B_1(0)} (1+|x|)^M |\partial^\alpha \phi| + \int_{[B_1(0)]^c} \frac{(1+|x|)^M |x|^{2n} |\partial^\alpha \phi|}{|x|^{2n}} \right]$$

$$\leq C \left[m\big(B_1(0)\big) \sup\{(1+|x|)^M |\partial^\alpha \phi|\} + \int_{[B_1(0)]^c} \frac{1}{|x|^{2n}} dx \cdot \sup\{(1+|x|)^M |x|^{2n} |\partial^\alpha \phi|\} \right].$$

The last quantity is finite since $\phi \in \mathcal{S}(\mathbb{R}^n)$ and thus the integral in (6.1) is indeed well defined. It's clear that (6.1) extends $\partial^\alpha f : C_c^\infty(\mathbb{R}^n) \to \mathbb{C}$ to $\mathcal{S}(\mathbb{R}^n)$. So all that's left to prove is that the linear functional defined by (6.1) is also continuous. By exactly the same computation as above, for some constant $C_2 > 0$ we obtain the estimate:

$$\left| (-1)^{|\alpha|} \int f \partial^\alpha \phi \right| \leq C_2 [\sup\{(1+|x|)^M |\partial^\alpha \phi|\} + \sup\{(1+|x|)^M |x|^{2n} |\partial^\alpha \phi|\}].$$

By expanding the $(1+|x|)^M$ via the binomial theorem and then doing some further estimates, we see that the above inequality implies that there exist $C_3 > 0$ and $N \in \mathbb{Z}_+$ such that

$$\left| (-1)^{|\alpha|} \int f \partial^\alpha \phi \right| \leq C_3 \sum_{|\beta| \leq N} \sup \left| x^\beta \partial^\alpha \phi \right|.$$

Thus the linear functional defined by (6.1) is indeed continuous. As discussed above, this proves that u is a tempered distribution.

Now let's prove the other direction. Suppose that $u \in \mathcal{D}'(\mathbb{R}^n)$ is a tempered distribution. We will assume that the support of u is contained in $\mathbb{R}_+^n = \{x \in \mathbb{R}^n : \text{each } x_k > 0\}$. The general case will then follow from the easy-to-see fact that any tempered distribution can be represented as a finite sum of translations and reflections of such tempered distributions and that reflections and translations of functions of polynomial growth are still of polynomial growth.

Alright, since u is a tempered distribution we have that there exist $C > 0$ and $N \in \mathbb{Z}_+$ such that

$$|\langle u, \phi \rangle| \leq C \sum_{|\alpha|, |\beta| \leq N} \sup \left| x^\alpha \partial^\beta \phi \right| \qquad \forall \phi \in \mathcal{S}(\mathbb{R}^n).$$

Let $E_{N+2} : \mathbb{R}^n \to \mathbb{C}$ denote the function

$$E_{N+2}(x) = \begin{cases} \dfrac{(x_1)^{N+1} \cdot \ldots \cdot (x_n)^{N+1}}{[(N+1)!]^n} & \text{if each } \ x_k \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Observe that E_{N+2} is in $C^N(\mathbb{R}^n)$ and that

$$\partial^{(N+2)\vec{1}} E_{N+2} = \delta$$

in the sense of distributions (here $\vec{1} \in \mathcal{I}(n)$ denotes the multi-index with all ones). Let $\rho \in C_c^\infty(\mathbb{R}^n)$ be a smooth function such that

1.) $\rho \geq 0$

2.) $\int \rho = 1$

3.) $\operatorname{supp} \rho \subseteq B_1(0)$.

For each $j \in \mathbb{Z}_+$, let $\rho_j \in C_c^\infty(\mathbb{R}^n)$ denote the functions $\rho_j = j^n \rho(jx)$. Observe that $\rho_j \to \delta$ in $\mathcal{D}'(\mathbb{R}^n)$. Now, we will prove that $E_{N+2} * u$ is a well-defined continuous function of polynomial growth such that

$$\partial^{(N+2)\vec{1}}(E_{N+2} * u) = u.$$

The fact that $E_{N+2} * u$ is well defined comes from the fact that the addition function $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ given by $(x, y) \mapsto x + y$ is clearly proper on $\overline{\mathbb{R}_+^n} \times \overline{\mathbb{R}_+^n}$ and hence proper on $\operatorname{supp} E_{N+2} \times \operatorname{supp} u$ (recall that $\operatorname{supp} E_{N+2}$ and $\operatorname{supp} u$ are both contained in $\overline{\mathbb{R}_+^n}$). The fact that the above equation holds follows immediately from

$$\partial^{(N+2)\vec{1}}(E_{N+2} * u) = \partial^{(N+2)\vec{1}}(E_{N+2}) * u = \delta * u = u.$$

Next let's show that $E_{N+2} * u$ is a continuous function. We will do this by showing that it's the uniform limit over compact sets of the continuous functions described in the following claim.

<u>Claim:</u> *For any $j \in \mathbb{Z}_+$, the following is a well-defined smooth function over $\mathbb{R}^n$:*

$$E_{N+2} * \rho_j * u.$$

Furthermore, if $\sigma \in C^\infty(\mathbb{R}^n)$ is such that $\sigma \equiv 1$ on a neighborhood of $supp(u)$ and $supp\, \sigma \subseteq \mathbb{R}_+^n$, then this function is explicitly given by

$$(6.2) \qquad E_{N+2} * \rho_j * u(x) = \langle u(y), \sigma(y)\big(E_{N+2} * \rho_j\big)(x - y)\rangle.$$

<u>Proof:</u> Fix any $j \in \mathbb{Z}_+$. The fact that the convolution $E_{N+2} * \rho_j * u$ is well defined follows from the not-hard-to-prove fact that the restriction of the addition function $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ given by $(x, y, z) \mapsto x + y + z$ to $\overline{\mathbb{R}_+^n} \times \overline{B_{1/j}(0)} \times \overline{\mathbb{R}_+^n}$ is proper (recall that $\operatorname{supp} \rho_j \subseteq \overline{B_{1/j}(0)}$). Now, let's see why the right-hand side of the above equation even makes sense. Specifically we have to show that argument of u on the right-hand side (i.e. $\sigma(y)\big(E_{N+2} * \rho_j\big)(x - y)$) is smooth and of compact support. The compactness of its support follows from:

$$\sigma(y)\big(E_{N+2} * \rho_j\big)(x-y) \neq 0 \quad \Longrightarrow \quad y \in \operatorname{supp}\sigma \quad \text{and} \quad (x-y) \in \operatorname{supp} E_{n+2} + \operatorname{supp}\rho_j$$

$$\Longrightarrow \quad y \in \mathbb{R}^n_+ \quad \text{and} \quad (x-y) \in \mathbb{R}^n_+ + B_{1/j}(0) \quad \Longrightarrow \quad y \in \mathbb{R}^n_+ \quad \text{and} \quad y \in x - \mathbb{R}^n_+ - B_{1/j}(0)$$

$$\Longrightarrow \quad |y| \leq |x| + 1/j.$$

The smoothness of the right-hand side of (6.2) also follows from this and Theorem 4.1.1 in the book.

To finish proving the claim, all we have to do now is show that equality holds in (6.2). To start, let $f = E_{N+2} * \rho_j$. Let $\sigma_f, \sigma_u \in C^\infty(\mathbb{R}^n)$ be such that

1.) $\sigma_f \equiv 1$ on a neighborhood of $\operatorname{supp}(f)$ and $\operatorname{supp}\sigma_f \subseteq \operatorname{supp}(f) + B_1(0)$,

2.) $\sigma_u \equiv 1$ on a neighborhood of $\operatorname{supp} u$ and $\operatorname{supp}\sigma_u \subseteq \operatorname{supp} u + B_1(0)$.

Take any test function $\phi \in C_c^\infty(\mathbb{R}^n)$. Analogously, let $\sigma_\phi \in C_c^\infty(\mathbb{R}^n)$ be such that $\sigma_\phi \equiv 1$ on a neighborhood of $\operatorname{supp}\phi$ (note the requirement for $\operatorname{supp}\sigma_\phi$ to be compact). By definition, we then have that

$$\langle f * u(z), \phi(z)\rangle = \langle f(x) \otimes u(y), \sigma_f(x)\sigma_u(y)\phi(x+y)\rangle$$

$$= \langle u(y), \int f(x)\sigma_f(x)\sigma_u(y)\phi(x+y)dx\rangle = \langle u(y), \int f(z-y)\sigma_u(y)\sigma_\phi(z)\phi(z)dx\rangle$$

$$= \langle u(y), \langle \phi(z), \sigma_u(y)\sigma_\phi(z)f(z-y)\rangle\rangle = \langle u(y) \otimes \phi(z), \sigma_u(y)\sigma_\phi(z)f(z-y)\rangle$$

$$= \int \phi(z)\langle u(y), \sigma_u(y)\sigma_\phi(z)f(z-y)\rangle dz = \langle\langle u(y), \sigma_u(y)f(z-y)\rangle, \phi(z)\rangle.$$

Since $\phi \in C_c^\infty(\mathbb{R}^n)$ was chosen arbitrarily, this shows that

$$f * u(z) = \langle u(y), \sigma_u(y)f(z-y)\rangle.$$

From here (6.2) follows immediately.

<u>End of Proof of Claim.</u>

Let $\sigma \in C^\infty(\mathbb{R}^n)$ be as in the above claim. First let's show that the sequence of functions $E_{N+2} * \rho_j * u$ is uniformly Cauchy over compact sets and thus converges to some continuous function. Fix any compact subset $K \subseteq \mathbb{R}^n$ and let $R > 0$ be such that $K \subseteq B_R(0)$. Then, by the above claim we have that

$$(6.3) \qquad \sup_{x\in K}\left|E_{N+2} * \rho_k * u(x) - E_{N+2} * \rho_j * u(x)\right|$$

$$= \sup_{x\in K}\left|\langle u(y), \sigma(y)\big[E_{N+2} * \rho_k(x-y) - E_{N+2} * \rho_j(x-y)\big]\rangle\right|$$

$$\leq C \sum_{|\alpha|,|\beta|\leq N} \sup_{x\in K}\sup_{y\in\mathbb{R}^n}\left|y^\alpha \partial_y^\beta\Big(\sigma(y)\big[E_{N+2} * \rho_k(x-y) - E_{N+2} * \rho_j(x-y)\big]\Big)\right|.$$

It's not hard to see that there exists a compact subset $Q \subseteq \mathbb{R}^n$ such that for any fixed $x \in K$, the support of the function

$$\sigma(y)\left[E_{N+2} * \rho_k(x-y) - E_{N+2} * \rho_j(x-y)\right]$$

is contained in Q (for instance, $Q = \overline{B_{R+1}(0)}$ will work). Because of this, we can change the "$y \in \mathbb{R}^n$" to "$y \in Q$" in the last supremum above without changing the value of the supremum. It's not hard to see then by the product rule that for some $\tilde{C} > 0$ the last quantity in (6.3) is further bounded by

$$\tilde{C} \sum_{|\gamma| \leq N} \sup_{x \in K} \sup_{y \in Q} \left|\partial_y^\gamma \left[E_{N+2} * \rho_k(x-y) - E_{N+2} * \rho_j(x-y)\right]\right|$$

$$\leq \tilde{C} \sum_{|\gamma| \leq N} \sup_{z \in K-Q} \left|\partial_y^\gamma \left[E_{N+2} * \rho_k(z) - E_{N+2} * \rho_j(z)\right]\right|.$$

Since $E_{N+2} \in C^N(\mathbb{R}^n)$, it's well known that for $\gamma \in \mathcal{I}(n)$ such that $|\gamma| \leq N$, $\partial^\gamma(E_{N+2} * \rho_k)$ converges uniformly to $\partial^\gamma E_{N+2}$ over compact sets (this is a slight variant of Theorem 1.2.1 in the book). Thus we see that the above quantity goes to zero as $k, j \to \infty$. So the first quantity in (6.3) also goes to zero as $k, j \to \infty$:

$$\sup_{x \in K} \left|E_{N+2} * \rho_k * u(x) - E_{N+2} * \rho_j * u(x)\right| \to 0 \qquad \text{as} \quad k, j \to \infty.$$

So indeed $E_{N+2} * \rho_j * u$ are uniformly Cauchy over compact sets and thus converge pointwise to some continuous function. I claim that that continuous function that they converge to is $E_{N+2} * u$. To see this, take any test function $\phi \in C_c^\infty(\mathbb{R}^n)$ and do (here I interchange "lim" and "$\int$," which I can do because of uniform convergence over compact sets)

$$\langle \lim_{j \to \infty} \left(E_{N+2} * \rho_j * u\right), \phi \rangle = \int \lim_{j \to \infty} \left(E_{N+2} * \rho_j * u(x)\right) \phi(x) dx$$

$$= \lim_{j \to \infty} \int E_{N+2} * \rho_j * u(x) \phi(x) dx = \lim_{j \to \infty} \langle E_{N+2} * \rho_j * u, \phi \rangle = \langle E_{N+2} * \delta * u, \phi \rangle$$

$$= \langle E_{N+2} * u, \phi \rangle.$$

Hence $E_{N+2} * u$ is indeed equal to the function that is the pointwise limit of $E_{N+2} * \rho_j * u$ and thus is a continuous function.

The last thing left to do is to show that $E_{N+2} * u$ is of polynomial growth. This is again just a game of bounding things. For any $x \in \mathbb{R}^n$ we have that

$$|E_{N+2} * u(x)| = \lim_{j \to \infty} \left|E_{N+2} * \rho_j * u(x)\right| = \lim_{j \to \infty} \left|\langle u(y), \sigma(y) \cdot E_{N+2} * \rho_j(x-y) \rangle\right|$$

$$\leq C \sum_{|\alpha|, |\beta| \leq N} \lim_{j \to \infty} \sup_{y \in \mathbb{R}^n} \left|y^\alpha \partial_y^\beta \left(\sigma(y) \cdot E_{N+2} * \rho_j(x-y)\right)\right|$$

As before, it's not hard to see that the support of the function $\sigma(y) \cdot E_{N+2} * \rho_j(x-y)$ as a function of y is contained in $\overline{\mathbb{R}^n_+} \cap \overline{B_{|x|+1}(0)}$. Thus we can change the "$y \in \mathbb{R}^n$" to "$y \in \overline{\mathbb{R}^n_+} \cap \overline{B_{|x|+1}(0)}$" in the above supremum without changing its value. By the product rule, it's then not hard to see that for some $C_2 > 0$ the above quantity can further be bounded by

$$C_2 \sum_{|\alpha|,|\beta| \leq N} \lim_{j\to\infty} \sup_{\substack{|y|\leq|x|+1 \\ \text{each } y_k \geq 0}} \left| y^\alpha \partial_y^\beta \left(E_{N+2} * \rho_j(x-y) \right) \right|.$$

Now, we have that $|y^\alpha| \leq (|x|+1)^{|\alpha|}$ for any y in the domain of the above supremum. In addition, as before, for any $\beta \in \mathcal{I}(n)$ such that $|\beta| \leq N$ we have that $\partial^\beta (E_{N+2} * \rho_j)$ converges to $\partial^\beta E_{N+2}$ uniformly over compact sets and so we can interchange the "limit" and "supremum" in the above quantity without changing its value. Thus we see that the above quantity is further bounded by

$$C_2(|x|+1)^{|\alpha|} \sum_{|\alpha|,|\beta| \leq N} \sup_{\substack{|y|\leq|x|+1 \\ \text{each } y_k \geq 0}} \left| \lim_{j\to\infty} \partial_y^\beta \left(E_{N+2} * \rho_j(x-y) \right) \right|$$

$$= C_2(|x|+1)^{|\alpha|} \sum_{|\alpha|,|\beta| \leq N} \sup_{\substack{|y|\leq|x|+1 \\ \text{each } y_k \geq 0}} \left| \partial_y^\beta E_{N+2}(x-y) \right|$$

$$= C_2(|x|+1)^{|\alpha|} \sum_{|\alpha|,|\beta| \leq N} \sup_{\substack{|y|\leq|x|+1 \\ \text{each } y_k \geq 0}} \left| \partial_y^\beta \left(\frac{(x-y)^{(N+1)\vec{1}}}{[(N+1)!]^n} \right) \right|$$

$$= \frac{C_2}{[(N+1)!]^n} (|x|+1)^{|\alpha|} \sum_{\substack{|\alpha|,|\beta| \leq N \\ \beta \leq (N+1)\vec{1}}} \sup_{\substack{|y|\leq|x|+1 \\ \text{each } y_k \geq 0}} \left| (x-y)^{(N+1)\vec{1}-\beta} \right|$$

(note the change under the last Σ symbol). Since each

$$|(x-y)|^{(N+1)\vec{1}-\beta} \leq (2|x|+1)^{|(N+1)\vec{1}-\beta|}$$

for all y in the domain of the above supremum, the last quantity in the previous equation is further bounded by

$$\frac{C_2}{[(N+1)!]^n} (|x|+1)^{|\alpha|} \sum_{\substack{|\alpha|,|\beta| \leq N \\ \beta \leq (N+1)\vec{1}}} (2|x|+1)^{|(N+1)\vec{1}-\beta|}.$$

Since this is a bound on $|E_{N+2} * u(x)|$, it is clear from here that $E_{N+2} * u$ is indeed of polynomial growth. This finally proves the theorem.

■

6.2 Poisson's Summation Formula [Theorem 8.5.1] (5/18/2021)

In this entry I work through the proof of the special case of the Poisson summation formula when the distribution in question is the Dirac delta function (Theorem 8.5.1 in the book) by putting it into my own words and filling in the details.

Theorem: *The following equality holds in $\mathcal{S}'(\mathbb{R}^n)$:*

$$(6.4)\qquad \sum_{g\in\mathbb{Z}^n}\tau_g\delta = \sum_{g\in\mathbb{Z}^n} e^{2\pi i g\cdot x},$$

where both sides are interpreted as limits of partial sums in $S'(\mathbb{R}^n)$ that exhaust $\mathbb{Z}^n$ in any way.

Remark: The above equation requires a bit of interpretation. The left-hand side of (6.4) denotes the functional $\mathcal{S}(\mathbb{R}^n)\to\mathbb{C}$ given by

$$(6.5)\qquad \phi\mapsto\sum_{g\in\mathbb{Z}^n}\phi(g)$$

while the right-hand side of (6.4) denotes the functional $\mathcal{S}(\mathbb{R}^n)\to\mathbb{C}$ given by

$$\phi\mapsto\sum_{g\in\mathbb{Z}^n}\hat{\phi}(2\pi g).$$

The claim of the theorem then is that these are well defined tempered distributions, the partial sums of $\sum_{g\in\mathbb{Z}^n}\tau_g\delta$ and $\sum_{g\in\mathbb{Z}^n}e^{2\pi i g\cdot x}$ converge to these two distributions in $\mathcal{S}'(\mathbb{R}^n)$ respectively, and that these two distributions are equal.

Proof: For brevity, let "u" denote the left-hand side of (6.4) and let "v" denote the right-hand side of (6.4). First let's show that that both u and v are well defined tempered distributions and that the partials sums on both sides of (6.4) converge to u and v respectively in $\mathcal{S}'(\mathbb{R}^n)$. One way to do this of course would be to say that this follows from the uniform boundedness principle and the fact that the limit of their respective partial sums' actions on members of $\mathcal{S}(\mathbb{R}^n)$ exist. But I will take a more elementary approach here by directly showing that both u and v are tempered distributions and that their respective partial sums converge to them in $\mathcal{S}'(\mathbb{R}^n)$.

Let's start with u. Observe that since any $\phi\in\mathcal{S}(\mathbb{R}^n)$ decays faster than any polynomial, we have the sum in (6.5) is absolutely convergent and thus the action of u on any $\phi\in\mathcal{S}'(\mathbb{R}^n)$ is well defined (and obviously linear). Next let's show that it's a tempered distribution. For any $\phi\in\mathcal{S}(\mathbb{R}^n)$ we have that (here $|g|$ denotes the usual Euclidian length of g)

$$|\langle u,v\rangle| = \left|\phi(0)+\sum_{g\in\mathbb{Z}^n\setminus\{0\}}\phi(g)\right| \le \sup|\phi| + \left(\sum_{g\in\mathbb{Z}^n\setminus\{0\}}\frac{1}{|g|^{2n}}\right)\sup\left||x|^{2n}\phi\right|.$$

Since $\sum_{g\in\mathbb{Z}^n\setminus\{0\}}1/|g|^{2n}$ is finite, this shows that $u:\mathcal{S}(\mathbb{R}^n)\to\mathbb{C}$ is indeed continuous and hence a tempered distribution. Finally, let's show that the partial sums on the left-hand side of (6.4) converge to u in $\mathcal{S}'(\mathbb{R}^n)$. Let $\{g_k\}_{k=1}^{\infty} \in \mathbb{Z}^n$ be any sequence that exhausts $\mathbb{Z}^n$ (i.e. the map $k \mapsto g_k$ is a bijective $\mathbb{Z}_+ \to \mathbb{Z}^n$ map). Then for any $\phi \in \mathcal{S}(\mathbb{R}^n)$ we have that

$$\lim_{m\to\infty} \langle \sum_{k=1}^{m} u_{g_k}, \phi \rangle = \lim_{m\to\infty} \sum_{k=1}^{m} \phi(g_k) = \sum_{g\in\mathbb{Z}^n} \phi(g) = \langle u, \phi \rangle.$$

Thus indeed $\sum_{k=1}^{m} u_{g_k} \to u$ in $\mathcal{S}'(\mathbb{R}^n)$.

Now let's show the same thing for v. The proof for showing that it's well defined and that the partial sums on the right-hand side of (6.4) converge to v in $\mathcal{S}'(\mathbb{R}^n)$ is done exactly the same way as for u by remembering that the Fourier transform maps $\mathcal{S}(\mathbb{R}^n)$ to itself. So I'll omit repeating those details and instead focus on showing that $v : \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}$ is continuous. Since the Fourier transform $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is continuous, we know that there exist $C_1, C_2 > 0$ and $N_1, N_2 \in \mathbb{Z}_+$ such that

$$\sup\left|\hat{\phi}\right| \leq C_1 \sum_{|\alpha|,|\beta|\leq N_1} \sup\left|x^{\alpha}\partial^{\beta}\phi\right|,$$

$$\sup\left||\xi|^{2n}\hat{\phi}\right| \leq C_2 \sum_{|\alpha|,|\beta|\leq N_2} \sup\left|x^{\alpha}\partial^{\beta}\phi\right|,$$

for all $\phi \in \mathcal{S}(\mathbb{R}^n)$. Thus, proceeding as with u, we have that for any $\phi \in \mathcal{S}(\mathbb{R}^n)$,

$$|\langle v, \phi \rangle| \leq \sup\left|\hat{\phi}\right| + \left(\sum_{g\in\mathbb{Z}^n\setminus\{0\}} \frac{1}{|g|^{2n}}\right) \sup\left||\xi|^{2n}\hat{\phi}\right|$$

$$\leq C_1 \sum_{|\alpha|,|\beta|\leq N_1} \sup\left|x^{\alpha}\partial^{\beta}\phi\right| + \left(\sum_{g\in\mathbb{Z}^n\setminus\{0\}} \frac{1}{|g|^{2n}}\right) C_2 \sum_{|\alpha|,|\beta|\leq N_2} \sup\left|x^{\alpha}\partial^{\beta}\phi\right|.$$

Hence indeed $v : \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}$ is continuous and thus is a tempered distribution.

Having dealt with these technical setup details, let's start proving the theorem. I claim that v is periodic over $\mathbb{Z}^n$:

$$v = \tau_h v \qquad\qquad \forall h \in \mathbb{Z}^n.$$

To see this, take any $h \in \mathbb{Z}^n$ and observe that

$$\tau_h v = \sum_{g\in\mathbb{Z}^n} \tau_h e^{2\pi i g\cdot x} = \sum_{g\in\mathbb{Z}^n} e^{2\pi i g\cdot h} e^{2\pi i g\cdot x} = \sum_{g\in\mathbb{Z}^n} e^{2\pi i g\cdot x} = v.$$

Now, let $\psi \in C_c^{\infty}(\mathbb{R}^n)$ be such that $\psi \geq 0$, $\operatorname{supp}\psi \subseteq (-1,1)^n$ and $\sum_{g\in\mathbb{Z}^n} \tau_g \psi = 1$ (the existence of such a ψ is proven in Lemma 8.5.1 in the book). I claim that

$$(6.6) \qquad v = \sum_{g \in \mathbb{Z}^n} (\tau_g \psi) v = \sum_{g \in \mathbb{Z}^n} \tau_g(\psi v)$$

in the sense that the partial sums here converge to v in $\mathcal{D}'(\mathbb{R}^n)$ (with respect to any exhaustion of $\mathbb{Z}^n$). These partial sums turn out to converge to v in $\mathcal{S}'(\mathbb{R}^n)$ as well, but we don't need that at the moment. Observe that the second equality here simply holds because of the fact that v is periodic over $\mathbb{Z}^n$. So let me prove the first equality. Take any $\phi \in C_c^\infty(\mathbb{R}^n)$. As before, let $\{g_k\}_{k=1}^\infty \in \mathbb{Z}^n$ be any sequence that exhausts $\mathbb{Z}^n$. Then we have that

$$\lim_{m \to \infty} \sum_{k=1}^m \langle (\tau_g \psi) v, \phi \rangle = \lim_{m \to \infty} \langle v, \sum_{k=1}^m (\tau_g \psi) \phi \rangle.$$

Since ϕ is of compact support, each $\operatorname{supp} \tau_g \psi \subseteq (-1,1)^n + g$, and $\sum_{g \in \mathbb{Z}^n} \tau_g \psi = 1$, we have that $\sum_{k=1}^m (\tau_g \psi) \phi$ is eventually equal to ϕ as $m \to \infty$. So the above limit is equal to $\langle v, \phi \rangle$ and thus indeed we have that the partial sums of $\sum_{g \in \mathbb{Z}^n} (\tau_g \psi) v$ converge to v in $\mathcal{D}'(\mathbb{R}^n)$.

Now, let's investigate what ψv is equal to. Focusing on v first, I claim that for any $j \in \{1, \dots, n\}$,

$$\left(e^{2\pi i x_j} - 1\right) v = 0.$$

To see this, observe that for any $j \in \{1, \dots, n\}$ we have that (here e_j is the standard unit vector with all zeros except a one in the j^{th} position)

$$e^{2\pi i x_j} v = \sum_{g \in \mathbb{Z}^n} e^{2\pi i x_j} e^{2\pi i g \cdot x} = \sum_{g \in \mathbb{Z}^n} e^{2\pi i (g + e_j) \cdot x} = \sum_{h \in \mathbb{Z}^n} e^{2\pi i h \cdot x} = v.$$

We obviously then have that for any $j \in \{1, \dots, n\}$,

$$(6.7) \qquad \left(e^{2\pi i x_j} - 1\right) \psi v = 0.$$

We can in fact write a better version of this equation by linearizing the leading coefficient:

<u>Claim:</u> Our ψv above satisfies the following: for any $j \in \{1, \dots, n\}$,

$$(6.8) \qquad x_j \psi v = 0.$$

<u>Proof:</u> Our approach will be to prove the above equality locally at any point. Fix any $j \in \{1, \dots, n\}$. Take any point $y \in \mathbb{R}^n$. First suppose that the function $\left(e^{2\pi i x_j} - 1\right)$ does not vanish at the point y (i.e. $y_j \notin \mathbb{Z}$). Choose a small enough neighborhood W of y such that $\left(e^{2\pi i x_j} - 1\right)$ does not vanish on W. Then equality in (6.8) over W follows from (6.7) by writing (6.7) as

$$\frac{e^{2\pi i x_j} - 1}{x_j} x_j \psi v = 0$$

and then dividing through by the nonzero $\left(e^{2\pi i x_j} - 1\right)/x_j$. Now suppose that it's the case that $\left(e^{2\pi i x_j} - 1\right)$ does vanish at y. If $y \neq 0$, then we can choose a neighborhood W of y such that

$0 \notin W$ and W is disjoint from ψ. Then (6.8) clearly holds on W because of the mere fact that $\psi \equiv 0$ on W. Lastly, suppose that $y = 0$. Let W be any neighborhood of y such that $W \subseteq (-1, 1)^n$. Then as before we can rewrite (6.7) over W as

$$\frac{e^{2\pi i x_j} - 1}{x_j} x_j \psi v = 0$$

where $\left(e^{2\pi i x_j} - 1\right)/x_j$ is interpreted to be equal to its limit value at $x = 0$, which observe makes it smooth. Since $\left(e^{2\pi i x_j} - 1\right)/x_j$ is nonzero over W, we again get that equality in (6.8) over W follows by dividing trough by $\left(e^{2\pi i x_j} - 1\right)/x_j$ in the above equation. Having proved the equality in (6.8) in a neighborhood of any point y in $\mathbb{R}^n$, the claim follows.

<u>End of Proof of Claim</u>

Now, take any $\phi \in C^\infty(\mathbb{R}^n)$. By Taylor's theorem we know that there exist $\sigma_j \in C^\infty(\mathbb{R}^n)$ for $j \in \{1, \dots, n\}$ such that

$$\phi(x) = \phi(0) + \sum_{j=1}^{n} x_j \sigma_j(x).$$

By (6.8) we then get that

$$\langle \psi v, \phi \rangle = \langle \psi v, \phi(0) \rangle = \langle \langle \psi v, 1 \rangle \delta, \phi \rangle.$$

Thus we've obtained that

$$\psi v = C\delta$$

for some constant $C \in \mathbb{C}$. By (6.6) we then get that

$$v = C \sum_{g \in \mathbb{Z}^n} \tau_g \delta \tag{6.9}$$

in $\mathcal{D}'(\mathbb{R}^n)$. Since we already proved that both v and $\sum_{g \in \mathbb{Z}^n} \tau_g \delta$ are tempered distributions, this equality in fact holds in $\mathcal{S}'(\mathbb{R}^n)$. Thus, the only task left to do in order to prove the theorem is to show that $C = 1$. To do this, let $\chi : \mathbb{R}^n \to \mathbb{C}$ denote the indicator function over $(0, 1)^n$:

$$\chi(x) = \begin{cases} 1 & \text{if } x \in (0,1)^n \\ 0 & \text{if } x \notin (0,1)^n \end{cases}.$$

Now let's take the convolution of both sides of (6.9) with χ. The left-hand side becomes

$$v * \chi = \sum_{g \in \mathbb{Z}^n} e^{2\pi i g \cdot x} * \chi = \sum_{g \in \mathbb{Z}^n} \begin{cases} 1 & \text{if } g = 0 \\ 0 & \text{if } g \neq 0 \end{cases} = 1.$$

Taking the convolution of the right-hand side of (6.9) with χ gives

$$\left(C\sum_{g\in\mathbb{Z}^n}\tau_g\delta\right)*\chi=C\sum_{g\in\mathbb{Z}^n}\big(\tau_g(\delta)*\chi\big)=C\sum_{g\in\mathbb{Z}^n}\Big(\delta*\tau_g(\chi)\Big)=C\sum_{g\in\mathbb{Z}^n}\tau_g\chi,$$

which is equal to the constant function C almost everywhere (with respect to the Lebesgue measure). Hence we must have that $C=1$. As discussed above, this proves the theorem.

∎

6.3 Fourier Transform of Tensor Products (Problem 8.3)

In this note we prove the following fact, which says that the Fourier transform of a tensor product is the tensor product of the Fourier transforms. This is obvious for L^1 functions; the tricky part is extending this to tempered distributions by duality.

Theorem 6.10: *Suppose that $u\in\mathcal{S}'(\mathbb{R}^n)$ and $v\in\mathcal{S}'(\mathbb{R}^m)$. Then $u\otimes v\in\mathcal{S}'(\mathbb{R}^{n+m})$ and*

$$(6.11)\qquad \widehat{u\otimes v}=\hat{u}\otimes\hat{v}.$$

Proof: First let's prove that the distribution $u\otimes v:C_c^\infty(\Omega)\to\mathbb{C}$ extends continuously to a linear map of the form $u\otimes v:\mathcal{S}(\mathbb{R}^n)\to\mathbb{C}$ and hence is also a tempered distribution. Let's do this by setting $u\otimes v$ to be the extension

$$(6.12)\qquad u\otimes v(\phi)=\langle u(x),\langle v(y),\phi(x,y)\rangle\rangle\qquad \forall\phi\in\mathcal{S}(\mathbb{R}^{n+m}),$$

and prove that this is well defined and continuous.

<u>Claim:</u> Let $f_\phi:\mathbb{R}^n\to\mathbb{C}$ be the function in the argument of u above:

$$f_\phi(x)=\langle v(y),\phi(x,y)\rangle.$$

Then $f_\phi\in\mathcal{S}(\mathbb{R}^n)$. Furthermore, the map $\phi\mapsto f_\phi$ is a continuous $\mathcal{S}(\mathbb{R}^{n+m})\to\mathcal{S}(\mathbb{R}^n)$ map.

<u>Proof:</u> Let's start by computing the partials of f_ϕ. Take any $j\in\{1,\dots,n\}$ and let $e_j=(0,\dots,0,1,0,\dots,0)$ be the standard unit vector with a "1" in the j^{th} place, treated as a multi-index here. I claim that

$$(6.13)\qquad \partial^{e_j}f_\phi(x)=\langle v(y),\partial_x^{e_j}\phi(x,y)\rangle.$$

We prove this by generalizing the argument in Theorem 4.1.1 in the book. For any $h\in\mathbb{R}:h\neq 0$ we have that

$$\left|\frac{f_\phi(x+he_j)-f_\phi(x)}{h}-\langle v(y),\partial_x^{e_j}\phi(x,y)\rangle\right|$$

$$=\left|\langle v(y),\frac{1}{h}\big[\phi(x+he_j,y)-\phi(x,y)\big]-\partial_x^{e_j}\phi(x,y)\rangle\right|.$$

Applying Taylor's theorem up to the first order term gives us that this is equal to

$$\left|\langle v(y), h\int_0^1 (1-t)\partial_x^{2e_j}\phi\big(x+the_j, y\big)dt\rangle\right|.$$

Applying the continuity of $v : \mathcal{S}(\mathbb{R}^m) \to \mathbb{C}$ tells us that for some $C > 0$ and some $M > 0$ the above quantity is bounded above by

$$C\sum_{|\alpha|+|\beta|\le M}\sup_{y\in\mathbb{R}^m}\left|y^\alpha\partial_y^\beta\left[h\int_0^1(1-t)\partial_x^{2e_j}\phi\big(x+the_j,y\big)dt\right]\right|$$

$$\le hC\sum_{|\alpha|+|\beta|\le M}\sup_{(x,y)\in\mathbb{R}^{n+m}}\left|y^\alpha\partial_x^{2e_j}\partial_y^\beta\phi(x,y)\right|.$$

This obviously goes to zero as $h \to 0$, and hence this proves (6.13). Furthermore, we see that for any $\lambda, \gamma \in I(n)$,

$$\left|x^\lambda\partial_\phi^\gamma f(x)\right| = \left|\langle v(y), x^\lambda\partial_x^\gamma\phi(x,y)\rangle\right| \le C\sum_{|\alpha|+|\beta|\le M}\sup_{y\in\mathbb{R}^m}\left|y^\alpha\partial_y^\beta\left[x^\lambda\partial_x^\gamma\phi(x,y)\right]\right|$$

$$\le C\sum_{|\alpha|+|\beta|\le M}\sup_{(x,y)\in\mathbb{R}^{n+m}}\left|x^\lambda y^\alpha\partial_x^\gamma\partial_y^\beta\phi(x,y)\right| < \infty.$$

So indeed $f \in S(\mathbb{R}^n)$. Furthermore, it's clear that this estimate shows that the mentioned map $\phi \mapsto f_\phi$ is continuous.

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Back to proving the theorem. Since we see that (6.12) is the composition of the continuous maps $u : \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}$ and $\phi \mapsto f_\phi$, we have that $u \otimes v$ is indeed a tempered distribution.

Now let's prove (6.11). Let's do this by proving that $\widehat{u \otimes v}$ and $\hat{u} \otimes \hat{v}$ agree on test functions of the form $\phi \otimes \psi$ where $\phi \in C_c^\infty(\mathbb{R}^n)$ and $\psi \in C_c^\infty(\mathbb{R}^m)$ as distributions since (6.11) will then follow by the density of finite linear combination of such test functions in all of $\mathcal{S}(\mathbb{R}^{n+m})$. For any $\phi \in C_c^\infty(\mathbb{R}^n)$ and $\psi \in C_c^\infty(\mathbb{R}^m)$ we have that

$$\langle\widehat{u\otimes v}, \phi\otimes\psi\rangle = \langle u\otimes v, \widehat{\phi\otimes\psi}\rangle = \langle u\otimes v, \hat{\phi}\otimes\hat{\psi}\rangle = \langle u(x), \langle v(y), \hat{\phi}(x)\hat{\psi}(y)\rangle\rangle$$

$$= \langle u, \hat{\phi}\rangle\langle v, \hat{\psi}\rangle = \langle\hat{u}, \phi\rangle\langle\hat{v}, \psi\rangle = \langle\hat{u}\otimes\hat{v}, \phi\otimes\psi\rangle.$$

As explained above, this proves the theorem.

■

7 Chapter 11

7.1 Wavefront Sets of Tensor Products

In this note I put the proof of Theorem 11.2.1 into my own words.

Theorem: *Suppose that $u \in \mathcal{D}'(\Omega)$ and $v \in \mathcal{D}'(\Theta)$ where $\Omega \subseteq \mathbb{R}^n$ and $\Theta \subseteq \mathbb{R}^m$ are open subset. Then*

$$WF(u \otimes v) \subseteq [WF(u) \times WF(v)] \cup [WF(u) \times (\operatorname{supp} v \times \{0\})] \cup [(\operatorname{supp} u \times \{0\}) \times WF(v)].$$

Remark: By writing $WF(u) \times WF(v)$, we are slightly abusing notation. What we really mean is the set

$$WF(u) \times WF(v)$$
$$= \{((x,y),(\xi,\eta)) \in (\Omega \times \Theta) \times (\mathbb{R}^{n+m} \setminus \{0\}) : (x,\xi) \in WF(u) \text{ and } (y,\eta) \in WF(v)\}.$$

A similar remark goes for the other two sets on the right-hand side of the previous equation.

Proof: Fix any point $((x_0,y_0),(\xi_0,\eta_0)) \in (\Omega \times \Theta) \times (\mathbb{R}^{n+m} \setminus \{0\})$. Let's take a look at what conditions on this point will imply that it's *not* in $WF(u \otimes v)$. For instance, we have that this point will not be in $WF(u \otimes v)$ if

$$\exists \phi \in C_c^\infty(\Omega) : \phi(x_0) \neq 0 \;\; \exists \psi \in C_c^\infty(\Theta) : \psi(y_0) \neq 0, \;\; (\xi_0,\eta_0) \notin \Sigma((\phi \otimes \psi)(u \times v)).$$

This condition here is furthermore implied by (here I use Theorem 6.10 from above):

$$(7.1) \qquad \exists \phi \in C_c^\infty(\Omega) : \phi(x_0) \neq 0 \quad \exists \psi \in C_c^\infty(\Theta) : \psi(y_0) \neq 0$$
$$\exists \text{ conic neighborhood } \Gamma \subseteq \mathbb{R}^{n+m} \setminus \{0\} \text{ of } (\xi_0,\eta_0) \;\; \forall N \in \mathbb{R} \;\; \exists C_N > 0,$$
$$|\widehat{\phi u}(\xi)||\widehat{\psi v}(\eta)| \leq C_N \langle (\xi,\eta) \rangle^{-N} \qquad \forall (\xi,\eta) \in \Gamma.$$

So let us ask ourselves, when can this condition hold? First, I claim that this condition holds if the following condition (A) holds:

$$(A) \qquad \xi_0 \neq 0 \text{ and } \eta_0 \neq 0, \text{ and } (x_0,\xi_0) \notin WF(u) \text{ and/or } (y_0,\eta_0) \notin WF(v).$$

To see why, suppose that (A) holds. Without loss of generality let's suppose that $(x_0,\xi_0) \notin WF(u)$. Then there exist $\phi \in C_c^\infty(\Omega) : \phi(x_0) \neq 0$ and a conic neighborhood $\tilde{\Gamma} \subseteq \mathbb{R}^n \setminus \{0\}$ of ξ_0 such that for all $N \in \mathbb{R}$ there exist $\tilde{C}_N > 0$ such that $|\widehat{\phi u}(\xi)| \leq \tilde{C}_N \langle \xi \rangle^{-N}$ for all $\xi \in \tilde{\Gamma}$. Observe also that since ψv is a distribution of compact support, it's Fourier transform $\widehat{\psi v}$ is of polynomial growth and hence there exists $M, C' > 0$ such that $|\widehat{\psi v}(\eta)| \leq C' \langle \eta \rangle^M$ for all $\eta \in \mathbb{R}^m$.

Consider the conic neighborhood $\Gamma = \tilde{\Gamma} \times \mathbb{R}^m \subseteq \mathbb{R}^{n+m} \setminus \{0\}$ of (ξ_0,η_0). Shrink Γ as a conic neighborhood around (ξ_0,η_0) if necessary to make the following quantity finite:

$$\alpha = \sup_{(\xi,\eta)\in\Gamma} \frac{|(\xi,\eta)|}{|\xi|} < \infty.$$

Then, we have that for any $N > 0$ and all $(\xi,\eta) \in \Gamma$,

$$\left|\widehat{\phi u}(\xi)\right|\left|\widehat{\psi v}(\eta)\right| \leq \tilde{C}_N C' \frac{1}{(1+|\xi|^2)^{N/2}}(1+|\eta|^2)^{M/2}$$

$$\leq \tilde{C}_N C' \frac{1}{\left(1+\frac{1}{\alpha^2}|(\xi,\eta)|^2\right)^{N/2}}(1+|(\xi,\eta)|^2)^{M/2}$$

$$\leq \begin{cases} \tilde{C}_N C' \alpha^2 \langle(\xi,\eta)\rangle^{-N+M} & \text{if } \alpha \geq 1 \\ \tilde{C}_N C' \langle(\xi,\eta)\rangle^{-N+M} & \text{if } \alpha \leq 1 \end{cases}.$$

It's easy to see that this implies (7.1).

Next, I claim that (7.1) holds if the following condition (B) holds:

(B) $\quad \xi_0 = 0$ and $\eta_0 \neq 0$, and $x_0 \notin \operatorname{supp} u$ and/or $(y_0, \eta_0) \notin WF(v)$.

This shouldn't be hard to see, and so I leave it to the reader to verify. Similarly, (7.1) holds if the following condition (C) holds:

(C) $\quad \xi_0 \neq 0$ and $\eta_0 = 0$, and $(x_0 \xi_0 \notin WF(u)$ and/or $y_0 \notin \operatorname{supp}(v)$.

Hence we have that a point satisfying either (A) or (B) or (C) is not in $WF(u \otimes v)$. Hence a point lying $WF(u \otimes v)$ does not satisfy A, does not satisfy (B), and does not satisfy (C). This is precisely what the equation in the theorem states.

■

7.2 Wavefront Set of Projection Pushforward

In this note, I put the proof of the following theorem that appears in the book into my own words by filling in the details. See Definition 2.13 for the definition of $\pi_* K$.

Theorem: *Suppose that $K \in \mathcal{D}'(X \times Y)$ is a distribution, where $X \subseteq \mathbb{R}^m$ and $Y \subseteq \mathbb{R}^n$ are open subsets, such that if $\pi : X \times Y \to X$ denotes the projection map $(x, y) \mapsto x$, then the restriction $\pi :$ supp $K \to X$ is a proper map. Then,*

$$\mathrm{WF}(\pi_* K) \subseteq \{(x,\xi) \in X \times (\mathbb{R}^n \setminus \{0\}) : \exists y \in Y \ \ (x, y, \xi, 0) \in \mathrm{WF}(K)\}.$$

Proof: Let's instead prove the equivalent statement that if $(x_0, \xi_0) \in X \times (\mathbb{R}^n \setminus \{0\})$ is such that for all $y \in Y$ we have that $(x_0, y, \xi_0, 0) \notin \mathrm{WF}(K)$, then $(x_0, \xi_0) \notin \mathrm{WF}(\pi_* K)$. Ok, suppose that we have such a point (x_0, ξ_0). Then, it's not hard to see that for any $y \in Y$ there exist $\phi \in C_c^\infty(X)$ and $\psi \in C_c^\infty(Y)$ such that $\phi \equiv 1$ and $\psi \equiv 1$ in bounded neighborhoods of x_0 and y respectively and $(x_0, y, \xi_0, 0) \notin \Sigma([\phi \otimes \psi]K)$. This last condition can of course be reformulated as that for some conic neighborhood $\Gamma \subseteq \mathbb{R}^{m+n} \setminus \{0\}$ of $(\xi_0, 0)$, for all $N \in \mathbb{R}$ there exists a $C_N > 0$ such that

$$|([\phi \otimes \psi]K)\hat{}\,(\xi,\eta)| \leq C_N \langle(\xi,\eta)\rangle^{-N} \qquad \forall (\xi,\eta) \in \Gamma.$$

Now, we have that the preimage of $\{x_0\}$ under the map $\pi : \operatorname{supp} K \to X$ is compact. Explicitly, this set is given by:

$$\operatorname{supp} K \cap (\{x_0\} \times Y) = \{x_0\} \times Q$$

for some compact set $Q \subseteq Y$. Because of Q's compactness, we can choose a finite number of points $y_j \in Y$ for $j = 1, \dots, l$ such that if ϕ_j, ψ_j, and Γ_j are as above to each y_j, then the regions over which $\psi_j \equiv 1$ cover Q. Let the following denote the following regions:

$$\Omega_j = \{x \in X : \phi_j(x) = 1\} \qquad \text{and} \qquad \Theta_j = \{y_j \in Y : \psi_j(y) = 1\}$$

which notice contain x_0 and y_j in the interiors respectively. Observe that by construction $Q \subseteq \bigcup_{j=1}^{l} \Theta_j$.

<u>Claim:</u> There exists a $\rho \in C_c^\infty(X)$ such that the following three conditions hold:

1. $\rho(x_0) \neq 0$,
2. $\operatorname{supp} \rho \subseteq \bigcap_{j=1}^{l} \Omega_j$
3.

$$\operatorname{supp} K \cap (\operatorname{supp} \rho \times Y) \subseteq \operatorname{supp} \rho \times \bigcup_{j=1}^{l} \Theta_j. \tag{7.2}$$

<u>Proof:</u> It's not hard to see that this claim will follow if we show that there exists a compact neighborhood $R \subseteq \bigcap_{j=1}^{l} \Omega_j$ of x_0 such that

$$\operatorname{supp} K \cap (R \times Y) \subseteq R \times \bigcup_{j=1}^{l} \Theta_j.$$

Let's prove this by contradiction: suppose not! Let $R_j \subseteq \bigcap_{j=1}^{l} \Omega_j$ for $j = 1,2,3, \dots$ be a sequence of compact balls centered at x_0 with radii tending to zero. Then by assumption we have that

$$\operatorname{supp} K \cap \left(R_j \times Y\right) \nsubseteq R_j \times \bigcup_{j=1}^{l} \Theta_j. \tag{7.3}$$

Notice that each set on the left-hand side here is compact since it's the preimage of R_j under the map $\pi : \operatorname{supp} K \to X$. We then have that (7.3) implies that for each j there exists a point $(x_j, z_j) \in \operatorname{supp} K \cap \left(R_j \times Y\right)$ such that $(x_j, z_j) \notin R_j \times \bigcup_{j=1}^{l} \Theta_j$. Since the sets on the left-hand side of (7.3) are contained in some fixed compact set, by passing to a subsequence if necessary, we may assume that (x_j, z_j) converges to a point of the form (x_0, z_0) for some $z_0 \in Y$. By the closedness of $\operatorname{supp} K$, this implies that

$$(x_0, z_0) \in \operatorname{supp} K \cap (\{x_0\} \times Y) = \{x_0\} \times Q \subseteq \{x_0\} \times \bigcup_{j=1}^{l} \Theta_j \subseteq R_j \times \bigcup_{j=1}^{l} \Theta_j \quad \forall j = 1,2,3, \ldots$$

But this then means that (x_j, z_j) is eventually in the sets on the right-hand side of (7.3), which is a contradiction. This proves the claim.

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Back to proving the theorem. Let $\rho \in C_c^\infty(X)$ be as in the above claim. Now, as in Definition 2.13, let $L \subseteq Y$ be a compact subset such that $\operatorname{supp} K \cap (\operatorname{supp} \rho \times Y) \subseteq \operatorname{supp} \rho \times L$ and let $\zeta \in C_c^\infty(Y)$ be such that $\zeta \equiv 1$ on a neighborhood of L. Furthermore, by (7.2) we can shrink L if necessary to ensure that it satisfies $L \subseteq \bigcup_{j=1}^{l} \Theta_j$. Let $R \subseteq Y$ be a compact set such that

$$L \subseteq R^{\text{int}} \subseteq R \subseteq \bigcup_{j=1}^{\infty} \Theta_j$$

and modify ζ so that it moreover satisfies $\operatorname{supp} \zeta \subseteq R$.

Let $\sigma_r \in C_c^\infty(Y)$ for $r = 1, \ldots, l$ and $\sigma_{l+1} \in C^\infty(Y)$ be a partition of unity subordinate to the open cover $\Theta_1, \ldots, \Theta_l, Y \setminus R$ of Y (in particular, we mean that each $\operatorname{supp} \sigma_i \subseteq \Theta_i$ and $\operatorname{supp} \sigma_{l+1} \subseteq Y \setminus R$). Then, for any $\xi \in \mathbb{R}^m \setminus \{0\}$ we have that

$$\widehat{\rho\pi_* K}(\xi) = \langle \rho\pi_* K, e^{-ix\cdot\xi} \rangle = \langle \pi_* K, \rho(x) e^{-ix\cdot\xi} \rangle = \langle K, \left[\rho(x) e^{-ix\cdot\xi}\right] \otimes \zeta(y) \rangle$$

$$= \langle [\rho \otimes \zeta] K, e^{-i(x,y)\cdot(\xi,0)} \rangle = ([\rho \otimes \zeta] K)^\wedge(\xi, 0) = \sum_{r=1}^{l+1} ([\rho \otimes (\zeta\sigma_r)] K)^\wedge(\xi, 0).$$

By construction we have that $\zeta\sigma_{l+1} \equiv 0$ and so in fact the last term $[\rho \otimes (\zeta\sigma_{l+1})]K$ in the last sum is equal to zero. From here we can conclude that for any $\xi \in \mathbb{R}^m \setminus 0$ such that $(\xi, 0) \in \bigcap_{j=1}^{l} \Gamma_j$ (note that this is a conic neighborhood of ξ_0) and all $N \in \mathbb{R}$,

$$\left|\widehat{\rho\pi_* K}(\xi)\right| \leq \sum_{r=1}^{l} |([\rho \otimes (\zeta\sigma_r)] K)^\wedge(\xi, 0)| \leq \sum_{r=1}^{l} |([\phi_r \otimes \psi_r] K)^\wedge(\xi, 0)| \leq C_N \langle (\xi, 0) \rangle^{-N}$$

for some constants $C_N > 0$. This shows that $\xi_0 \notin \Sigma(\rho\pi_* K)$ and hence indeed $(x_0, \xi_0) \notin \operatorname{WF}(\pi_* K)$. As discussed at the beginning of this proof, this proves the theorem.

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7.3 Lemma for Pullback of Wavefront sets by Diffeomorphisms

In this note I put the proof of the following result that appears in the book into my own words. It's important because it's a key step in demonstrating how the wavefront sets of distributions transform under change of coordinates.

Lemma: *Suppose that $F : U \to V$ is a diffeomorphism between open neighborhoods of 0 in $\mathbb{R}^n$ such that $F(0) = 0$ and $DF(0) = \mathrm{id}$ (i.e. the Jacobian at zero is the identity matrix). Then for any distribution $v \in \mathcal{D}'(V)$,*

$$\{(0,\xi) \in \mathrm{WF}(F^*v)\} = \{(0,\xi) \in \mathrm{WF}(v)\}.$$

Proof: Observe that by assumption

$$F^{-1}(y) = y + r(y)$$

for some remainder term $r \in O(y^2)$. Now, fix a distribution $v \in \mathcal{D}'(V)$. Take any direction $(0,\xi_0) \notin \mathrm{WF}(v)$. Let's show that $(0,\xi_0) \notin \mathrm{WF}(F^*v)$. Let $\psi_1, \psi_2 \in C_c^\infty(V)$ be compactly supported functions in V that will soon be determined and consider the distribution $F^*(\psi_1\psi_2 v)$. Its Fourier transform is given by

$$[F^*(\psi_1\psi_2 v)]^\wedge(\xi) = \langle F^*(\psi_1\psi_2 v)(x), e^{-ix\cdot\xi}\rangle = \langle \psi_1\psi_2 v(y), e^{-iF^{-1}(y)\cdot\xi}|\det F^{-1}(y)|\rangle$$

$$= \langle \psi_1 v(y), \psi_2 e^{-i(y+r(y))\cdot\xi}|\det F^{-1}|\rangle = \langle \widehat{\psi_1 v}(\eta), \int \psi_2(y) e^{i(y\cdot\eta - y\cdot\xi - r(y)\cdot\xi)}|\det F^{-1}|dy\rangle,$$

or in other words

$$(7.4) \qquad [F^*(\psi_1\psi_2 v)]^\wedge(\xi) = \iint \widehat{\psi_1 v}(\eta)\psi_2(y) e^{i(y\cdot\eta - y\cdot\xi - r(y)\cdot\xi)}|\det F^{-1}|dy d\eta.$$

Now, since $(0,\xi_0) \notin \mathrm{WF}(v)$ we can choose ψ_1 to be such that $\psi_1(0) \neq 0$ and there exists a conic neighborhood Γ such that for all $N \in \mathbb{R}$ there exists a $C_N > 0$ satisfying

$$\left|\widehat{\psi_1 v}(\eta)\right| \le C_N \langle\eta\rangle^{-N} \qquad \forall \eta \in \Gamma.$$

We will show that (7.4) is rapidly decreasing (i.e. decays faster than $|\xi|^{-k}$ for all $k > 0$) in a conic neighborhood of ξ_0 by separately showing that the $\eta \in \Gamma$ portion of the integral and $\eta \notin \Gamma$ portion have this property.

Ok, let's start with the integral portion over $\eta \in \Gamma$. Let $\phi = y \cdot \xi + r(y) \cdot \xi$ (appears in the exponent in (7.4)) and let $\mathrm{grad}_y\, \phi$ denote the gradient of ϕ with respect to y (while ξ is fixed). It's easy to see that

$$(7.5) \qquad \mathrm{grad}_y\, \phi = \left(I + \frac{\partial r_\mu}{\partial y_\nu}\right)\xi$$

where I denotes the square identity matrix and $\partial r_\mu/\partial y_\nu$ denotes the matrix with entries $\partial r_\mu/\partial y_\nu$ for $\mu, \nu = 1, \ldots, n$ (which notice are $O(y)$). Notice that if we define the following linear partial differential operator wherever $\mathrm{grad}_y\, \phi \neq 0$:

$$L = i\frac{1}{\left|\mathrm{grad}_y\, \phi\right|^2}\sum_{j=1}^n \frac{\partial\phi}{\partial y_j}\cdot\frac{\partial}{\partial y_j},$$

we then get that

$$(7.6) \qquad Le^{-i\phi(y,\xi)} = e^{-i\phi(y,\xi)}.$$

For our purposes notice that, by shrinking the support of ψ_2 if necessary while maintaining $\psi_2(0) \neq 0$, we can assume that $\operatorname{grad}_y \phi \neq 0$ over a neighborhood of $\operatorname{supp}\psi_2$. This is important since $\operatorname{supp}\psi_2$ is the effective dy integration domain in (7.4). Hence, if we plug in (7.6) into the right-hand side of (7.4) but when integration in η is only taken over Γ, and then integrate by parts in y, and then repeat this k number of times gives the quantity

$$\iint_{\eta\in\Gamma} L^T\left(\widehat{\psi_1 v}(\eta)\psi_2(y)e^{iy\cdot\eta}|\det F^{-1}|\right)e^{-i\phi(y,\xi)}dyd\eta$$

where L^T denotes the transpose of L. A little bit of thought should convince the reader that this quantity can be rewritten as

$$(7.7) \qquad \iint_{\eta\in\Gamma} e^{i(y\cdot\eta-y\cdot\xi-r(y)\cdot\xi)}\sum_{|\alpha|\leq k}\widehat{\psi_1 v}(\eta)\eta^\alpha f_{k,\alpha}(y,\xi)\,dyd\eta$$

for some smooth functions $f_{k,\alpha} : V \times (\mathbb{R}^n \setminus \{0\}) \to \mathbb{R}$ with $\operatorname{supp}_y f_{k,\alpha} \subseteq \operatorname{supp}\psi_2$ ("supp_y" means the projection of the support onto the "y" component). It's also not hard to see that since the coefficients of L are homogeneous of order -1 in ξ, we have that each $f_{k,\alpha}$ is homogeneous of order $-k$ in ξ. Hence, if we let

$$\lambda = \max_{|\alpha|\leq k}\sup_{\substack{y\in\operatorname{supp}\psi_2\\|\theta|=1}}\left|f_{k,\alpha}(y,\theta)\right| < \infty,$$

then (7.7) is bounded by (here $m_{\mathcal{L}}$ is the Lebesgue measure)

$$\left(\int_{\eta\in\Gamma}\sum_{|\alpha|\leq k}\widehat{\psi_1 v}(\eta)\eta^\alpha\,d\eta\right)m_{\mathcal{L}}(\operatorname{supp}\psi_2)\lambda k^k|\xi|^{-k},$$

where the important point here is that this is a finite constant times $|\xi|^{-k}$ (the $d\eta$ integral is finite since $\widehat{\psi_1 v}$ is rapidly decreasing in Γ). Hence the $\eta \in \Gamma$ portion of the integral in (7.4) is indeed rapidly decreasing (in all directions $\xi \in \mathbb{R}^n \setminus \{0\}$ in fact!).

Next let's look at the $\eta \notin \Gamma$ portion of the integral. Let $\sigma = y\cdot\eta - \phi(y,\xi)$ (also appears in the exponent in (7.4)), whose y gradient observe is equal to

$$\operatorname{grad}_y \sigma = I(\eta-\xi) - \frac{\partial r_\mu}{\partial y_\nu}\xi.$$

Shrinking the support of ψ_2 further if necessary while maintaining $\psi_2(0) = 0$, it shouldn't be difficult to see that there exists a conic neighborhood $\Gamma_1 \subseteq \overline{\Gamma_1} \subseteq \Gamma$ of ξ_0 which is small enough so that $\operatorname{grad}_y \sigma \neq 0$ for y in a neighborhood of $\operatorname{supp}\psi_2$, $\xi \in \overline{\Gamma_1}$ and $\eta \notin \Gamma$ (hint: let Γ_1 be the set of rays passing through a precompact neighborhood of $\xi_0/|\xi_0|$ in $\{|\xi| = 1\} \cap \Gamma$). Define the operator

$$M = -i\frac{1}{\left|\mathrm{grad}_y\, \sigma\right|^2}\sum_{j=1}^{n}\frac{\partial \sigma}{\partial y_j}\cdot\frac{\partial}{\partial y_j}$$

and observe that it satisfies

$$Me^{i\sigma(y,\eta,\xi)} = e^{i\sigma(y,\eta,\xi)}.$$

If we plug this into the right-hand side of (7.4) but when integrating only over $\eta \notin \Gamma$, and then integrate by parts in y, and then repeat this k number of times gives the following quantity for $\xi \in \Gamma_1$:

$$\iint_{\eta\notin\Gamma} M^T\left(\widehat{\psi_1 v}(\eta)\psi_2(y)|\det F^{-1}|\right)e^{i\sigma(y,\eta,\xi)}dyd\eta.$$

A little bit of thought should convince oneself that this quantity can be rewritten as

$$\iint_{\eta\notin\Gamma} \widehat{\psi_1 v}(\eta)g(y,\eta,\xi)e^{-i\sigma(y,\eta,\xi)}dyd\eta.$$

for some smooth function $g : V \times \Gamma^c \times \overline{\Gamma_1} \to \mathbb{R}$ such that $\operatorname{supp} g \subseteq \operatorname{supp}\psi_2$ ("Γ^c" means the complement of Γ in $\mathbb{R}^n \setminus \{0\}$). Notice that since the coefficients of M are homogeneous of order -1 in (η,ξ), g is homogeneous of order $-k$ in (η,ξ). Hence, if we let

$$\tilde{\lambda} = \sup_{\substack{y\in\operatorname{supp}\psi_2 \\ |(\theta,\omega)|=1}} |g(y,\theta,\omega)| < \infty,$$

then for big enough k the previous quantity is bounded by the following (i.e. for k big enough so that $|\eta|^k$ grows faster than $\widehat{\psi_2 v}(\eta)$ – which observe is indeed of polynomial growth))

$$m_{\mathcal{L}}(\operatorname{supp}\psi_2)\tilde{\lambda}\int_{\eta\notin\Gamma}\widehat{\psi_1 v}(\eta)\frac{1}{|(\eta,\xi)|^k}d\eta.$$

Let's break this integral up for the moment:

$$\int_{\substack{\eta\notin\Gamma \\ |\eta|\le 1}}\widehat{\psi_1 v}(\eta)\frac{1}{|(\eta,\xi)|^k}d\eta + \int_{\substack{\eta\notin\Gamma \\ |\eta|> 1}}\widehat{\psi_1 v}(\eta)\frac{1}{|(\eta,\xi)|^k}d\eta.$$

It's pretty clear that the first integral here decays like $|\xi|^{-k}$ for big enough $\xi \in \Gamma_1$. To see that the second integral has the same property, observe that using the elementary identity $(a-b)^2 \ge 0$, hence $a^2 + b^2 \ge 2ab$, and so $(a^2+b^2)^{-1} \le (2ab)^{-1}$, we see that this second integral is bounded by the following for sufficiently large $k > 0$:

$$\frac{m_{\mathcal{L}}(\operatorname{supp}\psi_2)\tilde{\lambda}}{2^{k/2}}\left(\int_{\substack{\eta\notin\Gamma\\|\eta|>1}}\widehat{\psi_1 v}(\eta)\frac{1}{|\eta|^{k/2}}\,d\eta\right)\frac{1}{|\xi|^{k/2}}.$$

Hence this shows that the $\eta \notin \Gamma$ portion of the integral in (7.4) is rapidly decreasing in the directions $\xi \in \Gamma_1$.

To summarize what we've proved: we've shown that for sufficiently large $k > 0$ there exists a constant $C_k > 0$ such that

$$|[F^*(\psi_1\psi_2 v)]^\wedge(\xi)| \le C_k\langle\xi\rangle^{-k} \qquad \forall\xi \in \Gamma_1.$$

Since $F^*(\psi_1\psi_2 v) = F^*(\psi_1\psi_2)F^*v$ and $F^*(\psi_1\psi_2) \in C_c^\infty(U)$ is nonzero at $x = 0$, this shows that indeed $(0,\xi_0) \notin \mathrm{WF}(F^*v)$. Since $(0,\xi_0) \notin \mathrm{WF}\, v$ was chosen arbitrarily, we've in fact demonstrated that

$$\{(0,\xi) \in \mathrm{WF}(F^*v)\} \subseteq \{(0,\xi) \in \mathrm{WF}(v)\}.$$

To show inclusion in the other direction, simply observe that by what we proved above but applied to F^{-1} gives:

$$\{(0,\xi) \in \mathrm{WF}(v)\} = \{(0,\xi) \in \mathrm{WF}((F^{-1})^*F^*v)\} \subseteq \{(0,\xi) \in \mathrm{WF}(F^*v)\}.$$

With this we've proved the lemma.

■