Haim's Notes About

Geometric Inverse Problems: with Emphasis on Two Dimensions

by Gabriel P. Paternain, Mikko Salo, Gunther Uhlmann

1 Table of Contents

2 Notations and Conventions

Convention 2.1: All manifolds are without boundary by default (i.e. unless stated otherwise).

Convention 2.2: All appropriate structures/maps are assumed to be smooth (C^∞) unless stated otherwise.

Convention 2.3: In a topological space, all neighborhoods are open unless stated otherwise.

Convention 2.4: The symbol $\mathbb{R}_+$ denotes the set of positive real numbers: $\mathbb{R}_+ = \{x \in \mathbb{R} : x > 0\}$.

Definition 2.5: If $P \subseteq M$ is an embedded submanifold of a Riemannian manifold without boundary (M, g), then U is called a **normal neighborhood** of P if it is a diffeomorphic image of a set of the form

$$\{(x, v) \in TM : v \perp P \text{ and } |v|_g < \delta(x)\},$$

for some smooth function $\delta : P \to (0, \infty)$, under the normal exponential map of P in M. It is called a **(ε-)tubular normal neighborhood** if $\delta \equiv \varepsilon$ for some $\varepsilon > 0$.

Definition 2.6: A **regular domain** is a smooth properly embedded submanifold with boundary of codimension zero.

Notation 2.7: If (M, g) is a Riemannian manifold possibly with boundary, $x \in M$ is a point on M, and $v \in T_xM$ is a tangent vector at x, then γ_v (or sometimes written $\gamma_{x,v}$) denotes the unique maximal geodesic with initial velocity v.

Notation 2.8: If (M, g) is a Riemannian manifold possibly with boundary, then SM denotes its unit sphere bundle.

Notation 2.9: Suppose that (M, g) is a Riemannian manifold with boundary and let $\nu : \partial M \to TM$ denote the inward pointing unit normal vector field at the boundary. Then the following denote:

$$\partial_+ SM = \{(x, w) \in \partial SM : \langle \nu, w \rangle \geq 0\},$$

$$\partial_0 SM = \{(x, w) \in \partial SM : \langle \nu, w \rangle = 0\},$$

$$\partial_- SM = \{(x, w) \in \partial SM : \langle \nu, w \rangle \leq 0\}.$$

The set $\partial_0 SM$ is also called the **glancing region**.

Definition 2.10: If (M, g) is a compact Riemannian manifold with boundary, then $[-\tau_-(x, v), \tau_+(x, v)]$ denotes the domain of $\gamma_{x,v}$ where $(x, v) \in SM$. We also write $\tau(x, v) = \tau_+(x, v)$, which is called the **exit time function**.

Definition 2.11: If (M, g) is a compact Riemannian manifold with boundary, then a **closed extension** $(N, \tilde{g})$ is a compact Riemannian manifold (without boundary) such that $M \subseteq N$ is a regular domain in N and $\tilde{g}$ smoothly extends g (i.e. the inclusion map $i : M \to N$ is an isometry). I typically let N be the double of M with $\tilde{g}$ being any smooth extension of g (see my notes on Jack Lee's book "Introduction to Riemannian Manifolds" for a proof of the existence of such a smooth extension).

Convention 2.12: If I put a hat "$\widehat{\ }$" over a variable, function, or set, it typically means that I'm working with its local coordinate version/representation. For example, if $f : M \to \mathbb{R}$ is a real valued function over a smooth manifold and (U, φ) is a chart of M, then $\hat{f} = f \circ \varphi^{-1}$. Since this convention is a relative imprecise statement, I will typically always define what my "hats" mean in my proofs below.

Notation 2.13: Suppose that (M,g) is an oriented two-dimensional Riemannian manifold and that $v \in T_xM$ is a tangent vector. We let $v^\perp$ denote the unique vector that is perpendicular to v, has the same length as v, and the ordered pair $(v, v^\perp)$ is positively oriented. We call $v^\perp$ the "90° rotation of v counterclockwise." Similarly, we let $v_\perp$ denote the same thing but instead we require that $(v, v^\perp)$ is negatively oriented. We call $v_\perp$ the "90° rotation of v clockwise." It's easy to see that the two are related by:

$$v_\perp = -v^\perp.$$

Definition 2.14: Suppose that (M,g) is an oriented 2-dimensional Riemannian manifold. Take any point $p \in M$ and a covector $\omega \in T_p^*M$ at p. We define the Hodge star operator "$\star$" applied to ω as the 90° rotation counterclockwise of ω. Precisely:

$$\star\, \omega = \left[\left(\omega^\sharp\right)^\perp\right]^\flat$$

where $\sharp$ and $\flat$ are the musical isomorphisms.

Definition 2.15: Suppose that (M,g) is an oriented 2-dimensional manifold. There are three standard vector fields over the geometry of the unit tangent bundle SM:

1.) We often let X denote the geodesic vector field over SM.

2) For any point $(x,v) \in SM$ and let $\psi_t(x,v) = (\gamma_{x,v^\perp}, W(t))$ where $W(t)$ is the parallel transport of v along $\gamma_{x,v^\perp}$ for some time. We often let $X_\perp = d/dt|_{t=0}\psi_t(x,v)$.

3.) Fix any point $(x,v) \in SM$ and consider the curve $\rho_t(x,v)$ given by $t \mapsto \cos(t)\, v + \sin(t)\, v^\perp$. We often let $V = d/dt|_{t=0}\rho_t(x,v)$. This vector field is called the **vertical derivative**.

Definition 2.16: Suppose that (M,g) is a compact nontrapping Riemannian manifold. We define the **scattering relation** $\alpha : \partial SM \to \partial SM$ to be the map

$$\alpha(x,v) = \begin{cases} \gamma_{x,v}\big(\tau(x,v)\big) & \text{if } (x,v) \in \partial_+ SM \\ (x,v) & \text{if } (x,v) \in \partial_0 SM \\ \gamma_{x,v}\big(-\tau(x,-v)\big) & \text{if } (x,v) \in \partial_- SM \end{cases}.$$

Definition 2.17: Suppose that (M,g) is a compact nontrapping Riemannian manifold and that $\mathcal{A} : SM \to \mathrm{GL}(n,\mathbb{C})$ is a smooth map. Let $U_+, U_- : SM \to \mathrm{GL}(n,\mathbb{C})$ be solutions to

$$\begin{cases} XU_+ + \mathcal{A}U_+ \\ U_+|_{\partial_- SM} = \mathrm{Id} \end{cases}$$

$$\begin{cases} XU_- + \mathcal{A}U_- \\ U_-|_{\partial_+ SM} = \mathrm{Id} \end{cases}$$

where X is the geodesic vector field on SM. We define the **scattering data**, or the **non-abelian x-ray transform**, of $\mathcal{A}$ to be the maps $C_{\mathcal{A}} = C_{\mathcal{A},+} : \partial_+ SM \to \mathrm{GL}(n, \mathbb{C})$ and $C_{\mathcal{A},-} : \partial_- SM \to \mathrm{GL}(n, \mathbb{C})$ given by

$$C_{A,+}(x, v) = U_+|_{\partial_+ SM},$$

$$C_{A,-}(x, v) = U_-|_{\partial_- SM}.$$

3 Chapter 3

3.1 Exercise 3.1.11 (Lemma 3.1.10)

Lemma: *Suppose that (M, g) is a compact Riemannian manifold with boundary and let $\nu : \partial M \to TM$ be the unit-normal inward pointing vector field along ∂M. Let $(N, \tilde{g})$ be a closed extension of M. Then there exists a smooth function $\rho : N \to \mathbb{R}$ such that*

1.) $\rho > 0$ on M, $\rho = 0$ on ∂M, $\rho < 0$ on M^c,

2.) $grad\, \rho = \nu$ on ∂M,

3.) $\rho(x) = d_g(x, \partial M)$ for $x \in M$ near M.

Proof: For any $\delta > 0$, let U_δ denote the δ-tubular normal neighborhood of ∂M in N if it exists. Since ∂M is compact, there does exist an ε-tubular normal neighborhood U_ε for some $\varepsilon > 0$. Let $E : U \subseteq N\partial M \to U_\varepsilon$ be the (diffeomorphic) normal exponential map of ∂M in N. Consider the diffeomorphism $F : W \subseteq \mathbb{R} \times \partial M \to U_\varepsilon$ given by:

$$F(t, x) = E\big(t\nu(x)\big)$$

where W is the following open set:

$$W = \{(t, x) : x \in \partial M, |t| < \varepsilon\}.$$

Let $\tilde{\rho} : V \to \mathbb{R}$ be the map given by $\tilde{\rho} = \pi_1 \circ F^{-1}$ where $\pi_1 : W \to \mathbb{R}$ is the projection $(x, t) \mapsto t$. This $\tilde{\rho}$ is of course smooth and will be the function that we want if we can extend it suitably onto all of N. Let $\phi, \psi_1, \psi_2 : N \to \mathbb{R}$ be smooth bump functions (i.e. $0 \le \phi, \psi_1, \psi_2 \le 1$) such that

1.) $\phi \equiv 1$ on $\overline{U_{3\varepsilon/4}}$ and $\operatorname{supp} \phi \subseteq U_\varepsilon$,

2.) $\psi_1 \equiv 1$ on $M \cap U^c_{2\varepsilon/4}$ and $\operatorname{supp} \psi_1 \subseteq M \cap \left(\overline{U_{\varepsilon/4}}\right)^c$,

3.) $\psi_2 \equiv 1$ on $M^c \cap U^c_{2\varepsilon/4}$ and $\operatorname{supp} \psi_2 \subseteq M^c \cap \left(\overline{U_{\varepsilon/4}}\right)^c$.

Then the smooth function $\rho : N \to \mathbb{R}$ given by

$$\rho = \phi \cdot \tilde{\rho} + \psi_1 - \psi_2$$

satisfies the conditions in the theorem and thus is the function that we want.

■

3.2 Lemma 3.1.12

Here I write down my own proof of the following lemma.

Lemma: *Suppose that (M, g) is a compact Riemannian manifold with boundary and let $\nu : \partial M \to TM$ be the unit-normal inward pointing vector field along ∂M. Let $(N, \tilde{g})$ be a closed extension of M and let $\rho : N \to \mathbb{R}$ be a function such as described in Exercise 3.1.11 (Lemma 3.1.10). Then for any $x \in \partial M$ and any $v \in T_x(\partial M)$,*

$$\frac{d}{dt}\rho \circ \gamma_v \bigg|_{t=0} = 0$$

and

$$-\Pi(v, v) = \nabla^2 \rho(v, v) = \frac{d^2}{dt^2}\rho \circ \gamma_v \bigg|_{t=0}.$$

Proof: The first equation is true because

$$\frac{d}{dt}\rho \circ \gamma_v \bigg|_{t=0} = \langle \operatorname{grad} \rho\,(x), \gamma_v'(0)\rangle = \langle \nu(x), v\rangle = 0.$$

Now let's prove the second equation. To prove the first equality there, we do:

$$-\Pi(v, v) = \langle D_t^{\gamma_v} \operatorname{grad} \rho\,, \gamma_v'(0)\rangle = \frac{d}{dt}\langle \operatorname{grad} \rho \circ \gamma_v, \gamma_v'\rangle \bigg|_{t=0} = \frac{d}{dt}(\rho \circ \gamma_v)'(t)|_{t=0}$$

$$= \frac{d^2}{dt^2}(\rho \circ \gamma_v)\bigg|_{t=0}.$$

To prove the second equality, in local coordinates we do:

$$\frac{d^2}{dt^2}\rho \circ \gamma_v \bigg|_{t=0} = \frac{d}{dt}\left(\dot{\gamma}^i(t)\frac{\partial \rho}{\partial x^i} \circ \gamma(t)\right)\bigg|_{t=0} = \ddot{\gamma}^i(0)\frac{\partial \rho}{\partial x^i}(x) + \dot{\gamma}^i(0)\dot{\gamma}^j(0)\frac{\partial^2 \rho}{\partial x^i \partial x^j}(x)$$

$$= -\Gamma_{ij}^k v^i v^j \frac{\partial \rho}{\partial x^k} + v^i v^j \frac{\partial^2 \rho}{\partial x^i \partial x^j} = \left(\frac{\partial^2 \rho}{\partial x^i \partial x^j} - \Gamma_{ij}^k \frac{\partial \rho}{\partial x^k}\right) v^i v^j = \nabla^2 \rho(v, v).$$

■

3.3 Lemma 3.2.3 and Exercise 3.2.4

Here I write out the proof of the following theorem with a bit more detail than in the book. Part of the reason is that I never learned the implicit function theorem on manifolds and hence this will give me an opportunity to work through such an application rigorously. In addition, the following proof will also contain the solution to Exercise 3.2.4.

Theorem: *Suppose that (M, g) is a compact non-trapping manifold with strictly convex boundary. Then the geodesic exit time function τ is continuous on SM and smooth on $SM \setminus \partial_0 SM$.*

Proof: Let's prove this by showing the following in the following order:

1.) τ is smooth and hence continuous on $\partial_- SM \setminus \partial_0 SM$,

2.) τ is smooth and hence continuous on $SM \setminus \partial_- SM$,

3.) τ is continuous at points of $\partial_0 SM$.

The statement in (1) is obvious since τ is constantly zero on $\partial_- SM \setminus \partial_0 SM$. So let's prove (2). Let $(N, \tilde{g})$ be a closed extension of (M, g) and let $\rho : N \to \mathbb{R}$ be a function as described in Exercise 3.1.11 (Lemma 3.1.10). Let $h : SN \times \mathbb{R} \to \mathbb{R}$ be the smooth function $h(x, v, t) = \rho \circ \gamma_{x,v}(t)$. Take any point $(x_0, v_0) \in SM \setminus \partial_- SM$. Observe that by Lemma 3.1.12 (seeing the fuller statement in the book might help) we have that

$$\frac{dh}{dt}\big(x, v, \tau(x, v)\big) = \langle \operatorname{grad} \rho \circ \gamma_{x,v}(t), \gamma'_{x,v}(t) \rangle\big|_{t=\tau(x,v)} < 0.$$

Here I've used the fact that $\gamma_{x,v}\big(\tau(x, v)\big)$ is on the boundary, which follows from compactness. Now, let (U, φ) be a chart of $SM \setminus \partial_- SM$ centered at (x_0, v_0). Consider the following local coordinate representation of h and τ respectively:

$$\hat{h} = h \circ (\varphi^{-1} \times \mathrm{id}_{\mathbb{R}}) \qquad \text{and} \qquad \hat{\tau} = \tau \circ \varphi^{-1}.$$

Let's furthermore agree on the convention that putting a hat "$\widehat{\ }$" over a point or set denotes its local coordinate representation (i.e. if $(x, v) \in U$ or $V \subseteq U$ then $(\hat{x}, \hat{v}) = \varphi(x, v)$ and $\hat{V} = \varphi[V]$). By the previous equation we then of course have that:

$$\frac{d\hat{h}}{dt}\big(\hat{x}_0, \hat{v}_0, \hat{\tau}(\hat{x}_0, \hat{v}_0)\big) < 0 \qquad \text{and} \qquad \hat{h}\big(\hat{x}_0, \hat{v}_0, \hat{\tau}(\hat{x}_0, \hat{v}_0)\big) = 0.$$

Now, I claim that τ is continuous near (x_0, v_0). Let's prove this by contradiction. Suppose not! Then there exists an $\varepsilon > 0$ and a sequence $\{(x_k, v_k)\}_{k=1}^{\infty}$ in SM such that $(x_k, v_k) \to (x_0, v_0)$ and each

$$|\tau(x_k, v_k) - \tau(x_0, v_0)| \geq \varepsilon.$$

Now, by the implicit function theorem there exists a radius $r > 0$ such that $B_r(\hat{x}_0, \hat{v}_0) \subseteq \widehat{U}$ (Euclidean ball), a $\delta > 0$, and a smooth function $\hat{f} : B_r(\hat{x}_0, \hat{v}_0) \to \mathbb{R}$ such that for any $(\hat{x}, \hat{v}) \in B_r(\hat{x}_0, \hat{v}_0)$, $\hat{f}(\hat{x}, \hat{y})$ is the unique number in $(\hat{\tau}(\hat{x}_0, \hat{y}_0) - \delta, \hat{\tau}(\hat{x}_0, \hat{y}_0) + \delta)$ such that

$$\hat{h}\left(\hat{x}, \hat{y}, \hat{f}(\hat{x}, \hat{y})\right) = 0.$$

By shrinking $r > 0$ if necessary, we can assume that $\delta < \varepsilon$. Furthermore, by cutting off the first few terms if necessary, we can assume that the sequence $\{(x_k, v_k)\}_{k=1}^{\infty}$ lies completely in $\varphi^{-1}[B_r(\hat{x}_0, \hat{y}_0)]$. Now, by definition of τ and the above equation we have that each $\tau(x_k, v_k) \leq f(x_k, v_k)$ (here $f = \hat{f} \circ \varphi$ since notice I "removed" the hat). Thus each:

$$\tau(x_k, v_k) \leq \tau(x_0, v_0) - \varepsilon.$$

More importantly, we get that the sequence of values $\tau(x_k, v_k)$ lie in the compact interval $[0, \tau(x_0, v_0) - \varepsilon]$. Thus be restricting to a subsequence if necessary, we can assume that $\tau(x_k, v_k)$ converges to some value $c \in [0, \tau(x_0, v_0) - \varepsilon]$ as $k \to \infty$. We then get that the sequence

$$\left(x_k, v_k, \tau(x_k, v_k)\right) \to (x_0, v_0, c) \qquad \text{as} \quad k \to \infty.$$

Since each

$$h\left(x_k, v_k, \tau(x_k, v_k)\right) = 0$$

and h is continuous, we then get that $h(x_0, v_0, c) = 0$. But this contradicts the definition of $\tau(x_0, v_0)$ since by construction $c < \tau(x_0, v_0)$. Thus τ must indeed be continuous at (x_0, v_0).

We're almost finished proving (2). The continuity of τ near (x_0, v_0) implies that there exists an open neighborhood $V \subseteq \varphi^{-1}[B_r(\hat{x}_0, \hat{v}_0)]$ of (x_0, v_0) such that:

$$\tau[V] \subseteq (\tau(x_0, v_0) - \delta, \tau(x_0, v_0) + \delta).$$

By the definition of f we then have that $\tau = f$ over V. Since f is smooth, we finally get that τ is indeed smooth near (x_0, v_0).

Finally let's prove (3). Take any $(x_0, v_0) \in \partial_0 SM$. Since M has strictly convex boundary, by Lemma 3.1.12 we have that $\tau(x_0, v_0) = 0$. By the same lemma we also have that

$$\frac{d^2 h}{dt^2}(x_0, v_0, 0) < 0.$$

$$\frac{dh}{dt}(x_0, v_0, 0) = 0.$$

Now, let $\alpha > 0$ be any positive number such that $\alpha < d^2h/dt^2\,(x_0, v_0, 0)$. By the continuity of d^2h/dt^2, we have that for any sufficiently small $\delta > 0$ there exists an open neighborhood $U \subseteq SM$ of (x_0, v_0) in SM such that

$$\frac{d^2h}{dt^2}(x, v, t) < -\alpha \qquad \forall (x, v, t) \in U \times (-\delta, \delta).$$

Let $\varepsilon > 0$ be any positive number (we'll determine what it should be later). By shrinking U if necessary, by the continuity of dh/dt we can assume that

$$\left|\frac{dh}{dt}(x, v, 0)\right| < \varepsilon \qquad \forall (x, v) \in U.$$

Great! We then have that for any $(x, v, t) \in U \times (-\delta, \delta)$,

$$h(v,t) = \int_0^t \frac{dh}{dt}(v,s)ds = \int_0^t \left[\frac{dh}{dt}(v,0) + \int_0^s \frac{d^2h}{dt^2}(v,r)dr\right] ds \leq \varepsilon t - \alpha\frac{t^2}{2} = \left(\varepsilon - \frac{\alpha}{2}t\right)t.$$

Thus if we chose our $\varepsilon > 0$ initially to be such that $\varepsilon < \alpha\delta/2$, then the above inequality shows that for any $t \in (-\delta, \delta)$ sufficiently close to δ the quantity $h(v, t)$ will be negative. Hence for any $(x, v) \in U$ we have that $\tau(x, v) < \delta$. Since $\delta > 0$ was arbitrary, this proves that τ is continuous at (x_0, v_0). Thus the theorem is finally proved.

■

3.4 (Definition 3.5.3) Vertical Vector Field in Coordinates

In Definition 3.5.3 in the book the authors define the vertical vector field V. In this section I derive an explicit expression for V in arbitrary coordinates. Let (x^i) be positively oriented coordinates of M and let (x^i, v^j) be the natural coordinates of TM given by $v^i(\partial/\partial x^i) \mapsto (x^i, v^i)$. Let g_{ij} denote the components of the metric tensor g in the coordinates (x^i). Now, in coordinate we have that

$$V = w^1\frac{\partial}{\partial w^1} + w^2\frac{\partial}{\partial w^2}$$

for some functions w^1, w^2. At any position (x^1, x^2, v^1, v^2) in these coordinates, I claim that w^1, w^2 satisfy the matrix equations

$$\begin{bmatrix} w^1 & w^2 \end{bmatrix}\begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0,$$

$$\begin{bmatrix} w^1 & w^2 \end{bmatrix}\begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = 1.$$

The first equation follows simply from the fact that V is perpendicular to the radial vector field $v^i(\partial/\partial v^i)$ with respect to the Sasaki metric (see Section 3.5 below), which is most easily seen by looking at the center of normal coordinates (alternatively, V must be perpendicular to the gradient of level set of the function $v \mapsto |v|_g^2$). The second equation simply follows from the fact that $|V|^2 \equiv 1$ with respect to the Sasaki metric (also most easily seen by looking at the center of normal coordinates). The above two equations can be rewritten as

$$w^1\left(g_{1j}v^j\right) + w^2\left(g_{i2}v^i\right) = 0,$$

$$g_{ij}w^iw^j = 1.$$

From the first equation we get that $(w^1, w^2) = \alpha(-g_{i2}v^i, g_{1j}v^j)$ for some scalar α. Plugging this into the second equation above gives that (here colors in each line indicate which terms combine together)

$$\alpha^2[g_{11}(g_{12}v^1 + g_{22}v^2)^2 - 2g_{12}(g_{12}v^1 + g_{22}v^2)(g_{11}v^1 + g_{12}v^2)$$
$$+g_{22}(g_{11}v^1 + g_{12}v^2)^2] = 1,$$

$$\alpha^2[g_{11}g_{12}^2(v^1)^2 + 2g_{11}g_{12}g_{22}v^1v^2 + g_{11}(g_{22})^2(v^2)^2$$
$$-2g_{12}^2g_{11}(v^1)^2 - 2(g_{12})^3v^1v^2 - 2g_{12}g_{22}g_{11}v^2v^1 - 2(g_{12})^2g_{22}(v^2)^2$$
$$+g_{22}(g_{11})^2(v^1)^2 + 2g_{22}g_{11}g_{12}v^1v^2 + g_{22}(g_{12})^2(v^2)^2] = 1,$$

$$\alpha^2[g_{11}(g_{22})^2(v^2)^2$$
$$-g_{12}^2g_{11}(v^1)^2 - 2(g_{12})^3v^1v^2 - (g_{12})^2g_{22}(v^2)^2$$
$$+g_{22}(g_{11})^2(v^1)^2 + 2g_{22}g_{11}g_{12}v^1v^2] = 1,$$

$$\alpha^2(g_{11}g_{22} - (g_{12})^2)[g_{22}(v^2)^2 + g_{11}(v^1)^2 + 2g_{12}v^1v^2] = 1,$$

$$\alpha = \sqrt{\frac{1}{\det g \cdot |v|_g^2}} = \frac{1}{\sqrt{\det g}}.$$

(we choose the positive square root at the end because V rotates around the unit circle counterclockwise in any oriented coordinate system). Hence we arrive at that

$$V = \frac{1}{\sqrt{\det g}}\left(-g_{i2}v^i \frac{\partial}{\partial v^1} + g_{1j}v^j \frac{\partial}{\partial v^2}\right).$$

3.5 Sasaki metric

On page 84 the authors define the Sasaki metric. Here is their definition but put into my own words:

Let (M,g) be a Riemannian n-manifold possibly with boundary and consider the natural projection $\pi : TM \to M$. Take any $p \in M$ and any $v \in T_pM$. Take any $\xi, \eta \in T_{(p,v)}TM$. Let $\alpha, \beta : I \to M$ be smooth curves and $V, W : I \to TM$ be smooth vectors along α and β respectively such that the curves $\Upsilon = (\alpha, V) : I \to TM$ and $\Theta = (\beta, W) : I \to TM$ satisfy

$$\Upsilon'(0) = \xi \qquad \text{and} \qquad \Theta'(0) = \eta.$$

Then, we define the inner product of ξ and η with respect to the **Sasaki metric** to be equal to

$$\langle \xi, \eta \rangle = \langle \alpha'(0), \beta'(0) \rangle_g + \left\langle \frac{DV}{dt}(0), \frac{DW}{dt}(0) \right\rangle_g$$

If (x^i) are coordinates of M and (x^i, v^j) are the natural coordinates of TM given by $v^i(\partial/\partial x^i) \mapsto (x^i, v^i)$, then this expression can also be written as (here $\partial_k = \partial/\partial x^k$)

$$\langle \xi, \eta \rangle = \left\langle \xi^k \partial_k, \eta^{\tilde{k}} \partial_{\tilde{k}} \right\rangle_g + \left\langle \left(\xi^{n+k} + \Gamma^k_{ij}\xi^i v^j\right)\partial_k, \left(\eta^{n+\tilde{k}} + \Gamma^{\tilde{k}}_{ij}\eta^i v^j\right)\partial_{\tilde{k}} \right\rangle_g$$

$$\langle \xi, \eta \rangle = g_{k\tilde{k}}\xi^k\eta^{\tilde{k}} + g_{k\tilde{k}}\left(\xi^{n+k} + \Gamma^k_{ij}\xi^i v^j\right)\left(\eta^{n+\tilde{k}} + \Gamma^{\tilde{k}}_{ij}\eta^i v^j\right).$$

(the fact that we are able to write this without any reference of α, β, V or W shows that this definition doesn't depend on them). Using notation from above, the **connection map** $K : TTM \to TM$ is defined as

$$K(\xi) = \frac{DV}{dt}(0) = \left(\xi^{n+k} + \Gamma^k_{ij}\xi^i v^j\right)\partial_k$$

Thus this allows us to write the Sasaki metric in a coordinate invariant way as:

$$\langle \xi, \eta \rangle = \langle d\pi(\xi), d\pi(\eta) \rangle_g + \langle K(\xi), K(\eta) \rangle_g.$$

If furthermore (x^i) are normal coordinates, then at the center of these coordinates

$$\langle \xi, \eta \rangle = \xi^1\eta^1 + \cdots + \xi^{2n}\eta^{2n}.$$

4 Chapter 11

4.1 Proposition 11.2.1

I want to make a comment about the proof of following theorem:

Theorem: *Suppose that M is a compact manifold with boundary and that g_1 and g_2 are two Riemannian metrics on it such that their distance functions are equal on the boundary: $d_{g_1} = d_{g_2}$ on $\partial M \times \partial M$. Then there exists a diffeomorphism $\psi : M \to M$ such that $\psi|_{\partial M} = id$ and*

$$g_1|_{\partial M} = \psi^* g_2|_{\partial M}$$

(precisely this means that for any $p \in \partial M$ and $v, w \in T_pM$, $g_1(v,w) = \psi^* g_2(v,w)$).

All I want to point out is that an alternative approach would be to use boundary flowouts of smooth extensions of the inward pointing unit normal vector fields along the boundary with respect to each metric g_1 and g_2.

4.2 Exercise 11.2.3

Here I prove the following result which is used in the proof of a theorem in this book.

Theorem: *Suppose that (M, g) is a Riemannian manifold with boundary. Let $p \in \partial M$ and $v \in T_p\partial M$. Then for any smooth curve $\gamma : (-\varepsilon, \varepsilon) \to \partial M$ such that $\gamma(0) = p$ and $\gamma'(0) = v$,*

$$|v|_g = \lim_{t \to 0^+} \frac{d_g\big(p, \gamma(t)\big)}{t}$$

where d_g denotes the Riemannian distance with respect to g of course.

Before we prove the above theorem, let's prove the following simpler version of the above result.

Lemma: *Suppose that (M, g) is a Riemannian manifold (without boundary). Let $p \in M$ and $v \in T_pM$. Then for any smooth curve $\gamma : (-\varepsilon, \varepsilon) \to M$ such that $\gamma(0) = p$ and $\gamma'(0) = v$,*

$$|v|_g = \lim_{t \to 0^+} \frac{d_g\big(p, \gamma(t)\big)}{t}.$$

Proof: Let $\big(U, (x^i)\big)$ be normal coordinates centered at p such that U is a geodesic ball. Then near $t = 0$ we have that (here $\gamma^i = x^i \circ \gamma$)

$$\lim_{t \to 0^+} \frac{d_g\big(p, \gamma(t)\big)}{t} = \lim_{t \to 0^+} \frac{1}{t}\sqrt{\big(\gamma^1(t)\big)^2 + \cdots + \big(\gamma^n(t)\big)^2} = \sqrt{\lim_{t \to 0^+}\left(\frac{\gamma^1(t)}{t}\right)^2 + \cdots + \lim_{t \to 0^+}\left(\frac{\gamma^n(t)}{t}\right)^2}$$

$$= \sqrt{\left(\dot{\gamma}^1(0)\right)^2 + \cdots + \left(\dot{\gamma}^n(0)\right)^2} = |\gamma'(0)|_g = |v|_g.$$

■

Now we are ready to prove the main result:

Proof of Theorem: Let $(N, \tilde{g})$ denote a Riemannian manifold (without boundary) that contains M as a regular domain and such that $\tilde{g}$ extends g smoothly (i.e. the inclusion $M \hookrightarrow N$ is an isometry). On one hand, notice that for any $t \in (-\varepsilon, \varepsilon) : t > 0$ we have that:

$$\frac{d_g\big(p, \gamma(t)\big)}{t} \leq \frac{\int_0^t |\gamma'(s)|_g}{t} = |\gamma'(s_t)|_g$$

where $0 \leq s_t \leq t$ is some number dependent on t. On the other hand, observe that $d_{\tilde{g}} \leq d_g$ over $M \times M$ since lengths of curves in M are the same with respect to $\tilde{g}$ and g but the definition of $d_{\tilde{g}}$ is an infimum over a larger set of curves than that of d_g (in particular those that go outside of M). Combining these two observations we get that

$$\frac{d_{\tilde{g}}\big(p, \gamma(t)\big)}{t} \leq \frac{d_g\big(p, \gamma(t)\big)}{t} \leq |\gamma'(s_t)|_g.$$

Now let's take the limit of all of these quantities as $t \to 0^+$. The limit of the rightmost quantity is $|\gamma'(0)|_g = |v|_g$ by the continuity of γ'. The previous lemma implies that the limit of the leftmost quantity is $|v|_{\tilde{g}} = |v|_g$. By the squeeze theorem we then get that indeed

$$\lim_{t \to 0^+} \frac{d_g\big(p, \gamma(t)\big)}{t} = |v|_g.$$

Hence the theorem is proved.

■

4.3 Proposition 11.2.5

Here I write down a proof of the following theorem with a bit (a lot!) more details than the one given in the book, with a bit of variation here and there. In my opinion, this theorem is a stunning result. Observe that even though we use Riemannian geometry to prove it, it's purely a statement about smooth manifolds.

Theorem: *Suppose that M is a connected and compact smooth n-manifold with boundary and that $\varphi : U \to U'$ is a diffeomorphism between two neighborhoods U and U' of the boundary ∂M such that $\varphi|_{\partial M} = id$. Then there exists a diffeomorphism $\psi : M \to M$ such that $\psi = \varphi$ near ∂M (i.e. in a neighborhood of ∂M).*

Proof: Let N be the double of M that contains M as a regular domain (note that N is compact as well). Let g be a Riemannian metric on M and let $\tilde{g}$ be a Riemannian metric on N that smoothly extends g (i.e. the inclusion $M \hookrightarrow N$ is an isometry). First let's extend φ to a neighborhood of ∂M that's open in N:

Claim: There exists a local diffeomorphism $\tilde{\varphi} : W \to N$ where W is a neighborhood of ∂M that's open in N such that $\tilde{\varphi} = \varphi$ over $W \cap M$.

Proof of Claim: Let V be a vector field over N that's normal to ∂M at points of ∂M. Consider the vector field $\varphi_* V$ over U'. Let W be an open neighborhood of ∂M such that $W \cap M \subseteq U \cap U'$. By shrinking W if necessary, we can assume that there exist diffeomorphic flowouts $\Phi_V : \mathcal{O} \to W$ and $\Phi_{\varphi_* V} : \mathcal{O} \to W$ of ∂M along V and $\varphi_* V$ respectively where $\mathcal{O}$ is the open set

$$\mathcal{O} = \{(t, p) : p \in \partial M, |t| < \delta(p)\} \subseteq \mathbb{R} \times \partial M$$

for some smooth positive real valued function $\delta : \partial M \to \mathbb{R}_+$ over ∂M. Now, let $\tilde{\varphi} : W \to N$ be the local diffeomorphism $\tilde{\varphi} = \Phi_{\varphi_* V} \circ \Phi_V^{-1}$. If we show that $\tilde{\varphi} = \varphi$ over $W \cap M$, then we'll have proved the claim. Take any $x_0 \in W \cap M$ and let $(t_0, p_0) = \Phi_V^{-1}(x_0)$. Let $\gamma, \sigma : \big(-\delta(p_0), \delta(p_0)\big) \to W$ denote the smooth curves:

$$\gamma(t) = \Phi_V(t, p_0) \qquad \text{and} \qquad \sigma(t) = \Phi_{\varphi_* V}(t, p_0).$$

We of course have that σ is an integral curve of $\varphi_* V$, but observe that $\varphi \circ \gamma$ is also an integral curve of $\varphi_* V$ because for any $t \in \operatorname{dom} \gamma$,

$$(\varphi \circ \gamma)'(t) = d\varphi_{\gamma(t)}\big(\gamma'(t)\big) = d\varphi_{\gamma(t)}\big(V_{\gamma(t)}\big) = (\varphi_* V)_{\varphi \circ \gamma(t)}.$$

Thus since σ and $\varphi \circ \gamma$ have the same starting point we have by the uniqueness of integral curves that

$$\tilde{\varphi}(x_0) = \Phi_{\varphi_* V} \circ \Phi_V^{-1}(x_0) = \Phi_{\varphi_* V}(t_0, p_0) = \sigma(t_0) = \varphi \circ \gamma(t_0) = \varphi \circ \Phi_V(t_0, p_0) = \varphi(x_0).$$

Hence the claim is proved!

End of proof of claim.

From now on, instead of the original φ that was given to us in the hypothesis of the theorem, let's let φ denote the smooth extension $\varphi : W \to N$ of φ constructed in the above claim where W is of course a neighborhood of ∂M that's open in N.

For any $r > 0$, let U_r denote the r-tubular normal neighborhood of ∂M in N if it exists (i.e. if it's the diffeomorphic image of the normal exponential map of ∂M in N). Ok, let's prove the following observation:

Claim: There exists an $r_0 > 0$ such that φ is defined over U_{r_0} and

$$\sup_{x\in U_{r_0}}\left\{d_{\tilde{g}}\big(x,\varphi(x)\big)\right\} < r_{\text{inj}}(N),$$

where $d_{\tilde{g}}$ denotes the distance function in $(N,\tilde{g})$ and $r_{\text{inj}}(N)$ is the injectivity radius of $(N,\tilde{g})$.

Proof of Claim: Since ∂M is compact, there exists an ε-tubular normal neighborhood U_ε for some $\varepsilon > 0$. Let $E : U \subseteq N\partial M \to U_\varepsilon$ be the (diffeomorphic) normal exponential map of ∂M in N. Now, let $r_0 > 0$ be such that $r_0 < \varepsilon$ and $\overline{U}_{r_0} \subseteq W$. Take any $x_0 \in U_{r_0}$. Let $t_0 > 0$ and $v_0 \in T_{p_0}M : |v_0|_{\tilde{g}} = 1$ and $v_0 \perp \partial M$ be the unique time and unit normal vector to ∂M such that $E(t_0 v_0) = x_0$. Let $\gamma : [0,r_0] \to U_\varepsilon$ denote the unit-speed geodesic given by $\gamma(t) = E(tv_0)$ which notice satisfies the properties that $\gamma(0) = p_0$, $\gamma(t_0) = x_0$, and $L_{\tilde{g}}(\gamma) < r_0$ where "$L_{\tilde{g}}$" denotes the "length of curve" operator. We then have that (recall that $\varphi|_{\partial M} = \text{id}$):

$$d_{\tilde{g}}\big(x_0,\varphi(x_0)\big) \le d_{\tilde{g}}(x_0,p_0) + d_{\tilde{g}}\big(p_0,\varphi(x_0)\big) = d_{\tilde{g}}(x_0,p_0) + d_{\tilde{g}}\big(\varphi(p_0),\varphi(x_0)\big)$$

$$< r_0 + L_{\tilde{g}}(\varphi\circ\gamma) \le r_0 + \|d\varphi\|_{L^\infty(\overline{U}_{r_0});\tilde{g}} L_{\tilde{g}}(\gamma) < \left(1 + \|d\varphi\|_{L^\infty(\overline{U}_{r_0});\tilde{g}}\right) r_0$$

where

$$\|d\varphi\|_{L^\infty(\overline{U}_{r_0});\tilde{g}} = \sup_{v\in T_xN : x\in\overline{U}_{r_0} \text{ and } |v|_{\tilde{g}}=1}\left\{|d\varphi_x(v)|_{\tilde{g}}\right\}.$$

From here we see that it's possible shrink $r_0 > 0$ sufficiently so as to satisfy the conclusion of the claim.

End of proof of claim.

Now, for any $x \in U_{r_0}$ let $\eta_x : [0,1] \to N$ denote the unique minimizing geodesic in N such that $\eta_x(0) = x$, $\eta_x(1) = \varphi(x)$. Observe that the curve η_x constantly stays inside the Riemannian ball

$$B_{r_{\text{inj}}(N)}(x) = \left\{y \in N : d_{\tilde{g}}(y,x) < r_{\text{inj}}(N)\right\}$$

where recall that $\exp_x$ maps diffeomorphically onto this set. Observe also that we can explicitly write this function $\eta : [0,1] \times U_{r_0} \to N$ as:

$$\eta_x(t) = \exp\left(t\exp_x^{-1}\big(\varphi(x)\big)\right).$$

Let's prove that η is smooth:

Claim: The map η is smooth.

Proof of Claim: Let's prove that η is smooth by working inside out in the above explicit equation for η. Let $\mathcal{O}$ denote the following open subset of $N \times N$:

$$\mathcal{O} = \left\{(x,y) \in N \times N : d_{\tilde{g}}(x,y) < r_{\text{inj}}(N)\right\},$$

and consider the function $F : \mathcal{O} \to TN$ given by:

$$F(x,y) = \exp_x^{-1}(y).$$

Let's prove that F is smooth. Take any $(x_0, y_0) \in \mathcal{O}$ and let $v_0 \in T_{x_0}N : |v_0|_{\tilde{g}} < r_{\text{inj}}(N)$ be such that $\exp(x_0, v_0) = y_0$. Let's set up a local coordinate representation of exp near (x_0, v_0). Let $(\mathcal{U}, \phi)$ and $(\mathcal{V}, \psi)$ be smooth charts of N and (E_i) be a smooth orthonormal frame of TN over $\mathcal{U}$. Let $\pi : TN \to N$ denote the natural projection map $(x, v) \mapsto x$ and let $\Omega \subseteq \pi^{-1}[\mathcal{U}]$ be a neighborhood of (x_0, v_0) such that

$$\exp[\Omega] \subseteq \mathcal{V}.$$

We do this of course since if we then take the smooth chart $\left(\Omega, \tilde{\phi}\right)$ of TN given by

$$\tilde{\phi}\left(x, v^i E_i\right) = (\phi(x), v^1, \dots, v^n),$$

we then have the following local coordinate representation of exp to work with:

$$\widehat{\exp} = \psi \circ \exp \circ \tilde{\phi}^{-1}.$$

Now, let us adopt the convention here that putting a hat over a point in Ω denotes its local coordinate representation in $\tilde{\phi}$ (i.e. $(\hat{x}, \hat{v}) = \tilde{\phi}(x, v)$). Our ambition is to apply the implicit function theorem on $\widehat{\exp}$ in the variable $\hat{v}$ near $(\hat{x}_0, \hat{v}_0)$. That will give us a smooth local coordinate representation of F near (x_0, v_0) and thus prove that F is indeed smooth. To do this task, we first need to show that the square matrix with columns $\partial\widehat{\exp}/\partial\hat{v}^i\,(\hat{x}_0, \hat{v}_0)$ for $i = 1, \dots, n$ is invertible. For any $i = 1, \dots, n$ let e_i denote the row matrix $[0, \dots, 0, 1, 0, \dots, 0]$ with all zeros except a 1 in the i^{th} entry. Before we proceed to the computation, let's observe the additional fact that the above charts also give us the following local coordinate representation of $\exp_{x_0} : T_{x_0}N \to M$ given by:

$$\widehat{\exp_{x_0}}(\hat{x}, \hat{v}) = \psi \circ \exp_{x_0}\left(\hat{v}^\mu E_\mu\big|_{x_0}\right).$$

With this in hand we do

$$\frac{\partial\widehat{\exp}}{d\hat{v}^i}(x_0, v_0) = \frac{d}{dt}\bigg|_{t=0}\left(\psi \circ \widehat{\exp} \circ \tilde{\phi}^{-1}(t \mapsto [\hat{x}_0, \hat{v}_0 + te_i])\right)$$

$$= \frac{d}{dt}\bigg|_{t=0}\left(\psi \circ \exp_{x_0}\left(t \mapsto \left[\hat{v}_0^\mu E_\mu + tE_i\right]\right)\right) = \frac{d}{dt}\bigg|_{t=0}\left(\widehat{\exp_{x_0}}(\hat{v}_0 + te_i)\right) = \frac{\partial\widehat{\exp_{x_0}}}{\partial\hat{v}^i}(\hat{v}_0).$$

Hence, we arrive at the equation

$$\frac{\partial\widehat{\exp}}{d\hat{v}^i}(x_0, v_0) = \frac{\partial\widehat{\exp_{x_0}}}{\partial\hat{v}^i}(\hat{v}_0) \qquad \text{for} \quad i = 1, \dots, n.$$

(some might argue that this equation is obvious). Since $\exp_{x_0}$ is diffeomorphic onto $B_{r_{\text{inj}}(N)}(x_0)$, the column vectors on the right-hand side of this equation are linearly independent. So indeed the square matrix with columns $\partial\widehat{\exp}/\partial\hat{v}^i\,(\hat{x}_0,\hat{v}_0)$ for $i = 1, \ldots, n$ is invertible. Thus by the implicit function theorem there exist Euclidean balls $\hat{B}_{\rho_1}(\hat{x}_0) \subseteq \mathcal{U}$ and $\hat{B}_{\rho_2}(\hat{y}_0) \subseteq \mathcal{V}$ centered at $\hat{x}_0$ and $\hat{y}_0$ in $\mathcal{U}$ and $\mathcal{V}$ of radius ρ_1 and ρ_2 respectively and a smooth function $\hat{F} : \hat{B}_{\rho_1}(\hat{x}_0) \times \hat{B}_{\rho_2}(\hat{y}_0) \to \mathbb{R}^n$ such that

$$\widehat{\exp}\left(\hat{x}, \hat{F}(\hat{x},\hat{y})\right) = \hat{y} \qquad \text{for} \;\; \forall x \in \hat{B}_{\rho_1}(\hat{x}_0) \;\; \forall y \in \hat{B}_{\rho_2}(\hat{y}_0).$$

It’s not hard to see that our function $\hat{F}$ here is a local coordinate representation of F. Thus we’ve proven that F is smooth near (x_0, y_0) and hence is smooth everywhere. The fact that η is smooth then immediately follows from the equation

$$\eta(t,x) = \exp tF\big(x, \varphi(x)\big)$$

and the fact that compositions of smooth functions is smooth.

End of proof of claim

Next we prove the following claim:

Claim: There exist positive numbers $r, s > 0$ such that for any $t \in [0,1]$, the map $x \mapsto \eta_x(t)$ is a diffeomorphism from U_r onto a neighborhood of U_s (we must of course have that $r \leq r_0$).

Proof of Claim: In order to study the maps in question, let’s give them notation: for any $t \in [0,1]$, let $F_t : U_{r_0} \to N$ denote the smooth map given by $x \mapsto \eta_x(t)$. Let’s first show that the differential of F_t is not singular at any point of ∂M. As before, let $\mathcal{O}$ denote the following open subset of $N \times N$:

$$\mathcal{O} = \left\{(x,y) \in N \times N : d_{\tilde{g}}(x,y) < r_{\text{inj}}(N)\right\}$$

and notice that we can write F_t as the composition

$$F_t = H_t \circ \left(\text{id}_{U_{r_0}} \times \varphi\right)$$

where $H_t : \mathcal{O} \to N$ is the map:

$$H_t(x,y) = \exp\left(t\exp_x^{-1}(y)\right).$$

The fact that H_t is smooth for all fixed $t \in [0,1]$ follows from the proof of the previous claim. Observe that H_t also satisfies the following property by the uniqueness of geodesics:

$$H_t(x,y) = H_{1-t}(y,x).$$

Now, take any time $t \in [0,1]$ and any point $x \in \partial M$. We want to compute $d(F_t)_x$. I'm going to do this *very* carefully. Take any $v \in T_xN$. We then have that (observe that $\varphi(x) = x$ since $\varphi|_{\partial M} = \mathrm{id}$)

$$d(F_t)_x(v) = d\left(H_t \circ \left(\mathrm{id}_{U_{r_0}} \times \varphi\right)\right)_x(v) = d(H_t)_{(x,\varphi(x))}\big(v, d\varphi_x(v)\big)$$

$$= d(H_t)_{(x,x)}(v,0) + d(H_t)_{(x,x)}\big(0, d\varphi_x(v)\big).$$

Let's compute the first vector on the right-hand side here. We have that (here γ_v denotes the maximal geodesic in N with starting velocity v)

$$d(H_t)_{(x,x)}(v,0) = \left.\frac{d}{ds}\right|_{s=0} H_t(\gamma_v(s), x) = \left.\frac{d}{ds}\right|_{s=0} H_{1-t}\big(x, \gamma_v(s)\big)$$

$$= \left.\frac{d}{ds}\right|_{s=0} \exp\left((1-t)\exp_x^{-1}\big(\gamma_v(s)\big)\right) = \left.\frac{d}{ds}\right|_{s=0} \exp\big((1-t)sv\big)$$

$$= \left.\frac{d}{ds}\right|_{s=0} \gamma_{(1-t)v}(s) = (1-t)v.$$

Now let's compute the second vector on the right-hand side of the previous equation:

$$d(H_t)_{(x,x)}\big(0, d\varphi_x(v)\big) = \left.\frac{d}{ds}\right|_{s=0} \left(H_t(x, \gamma_{d\varphi_x(v)}(s)\right) = \left.\frac{d}{ds}\right|_{s=0} \exp\left(t\exp_x^{-1}\left(\gamma_{d\varphi_x(v)}(s)\right)\right)$$

$$= \left.\frac{d}{ds}\right|_{s=0} \exp\big(tsd\varphi_x(v)\big) = \left.\frac{d}{ds}\right|_{s=0} \gamma_{td\varphi_x(v)}(s) = td\varphi_x(v).$$

Hence from the second to last equation we finally get that

$$d(F_t)_x(v) = (1-t)v + td\varphi_x(v).$$

Thus we've computed that the differential of F_t at x is given by

$$d(F_t)_x = (1-t)\mathrm{id}_{T_xN} + td\varphi_x.$$

To show that this is nonsingular, take any orthonormal basis (v_i) of T_xN such that $v_1, \ldots, v_{n-1}$ span $T_x\partial M$ and $v_n \perp \partial M$ is inward pointing. Since $\varphi|_{\partial M} = \mathrm{id}$, we have that $d\varphi_x(v_i) = v_i$ for $i = 1, \ldots, n-1$. Furthermore, since $d\varphi_x$ is nonsingular and φ maps points of M to M we must have that $d\varphi_x(v_n)$ has a positive component in the direction v_n:

$$\langle d\varphi_x(v_n), v_n\rangle_{\tilde{g}} > 0.$$

Thus $d(F_t)(v_n)$ has positive component in the direction v_n:

$$\langle dF_t(v_n), v_n\rangle_{\tilde{g}} = (1-t) + t\langle d\varphi_x(v_n), v_n\rangle_{\tilde{g}} > 0.$$

So the list of vectors

$$d(F_t)(v_i) = \begin{cases} v_i & \text{if } \ i = 1, \dots, n-1 \\ (1-t)v_n + td\varphi_x(v_n) & \text{if } \ i = n \end{cases}.$$

is linearly independent, which then of course implies that $d(F_t)_x$ is indeed nonsingular.

Great! With this in hand, I claim that there exists an $r > 0$ such that $d(F_t)$ is nonsingular over U_r for all $t \in [0,1]$. To see why, suppose that this wasn't the case. Then there would exist a sequences $\{t_k\}_{k=1}^{\infty} \subseteq [0,1]$ and $\{x_k\}_{k=1}^{\infty} \subseteq U_{r_0}$ such that each $d\left(F_{t_k}\right)_{x_k}$ is singular and $d_{\tilde{g}}(x_k, \partial M) \to 0$. Since $[0,1]$ and N are compact, by restricting to subsequences if necessary, we can assume that $t_k \to t_0$ and $x_k \to x_0$ for some limits $t_0 \in [0,1]$ and $x_0 \in N$. Now, the continuity of the distance function $x \mapsto d_{\tilde{g}}(x, \partial M)$ implies that $x_0 \in \partial M$. Similarly, the continuity of F implies that $d\left(F_{t_0}\right)_{x_0}$ is singular (most easily seen in local coordinates). But that contradicts the fact that we proved right before: that $d(F_t)_x$ is nonsingular at any $x \in \partial M$ for all $t \in [0,1]$. Thus there indeed exists an $r > 0$ such that $d(F_t)$ is nonsingular over U_r.

Next let's prove that there exists an $r > 0$ such that F_t is injective over U_r for all $t \in [0,1]$ (different r now). To do this we will need to utilize the smooth map $G : [0,1] \times U_{r_0} \to \mathbb{R} \times N$ given by

$$G(t,x) = \big(t, F(t,x)\big).$$

Let's make a few observations about this map. It's not hard to see from this equation that for any $t \in [0,1]$ and any $x \in \partial M$, $dG_{(t,x)}$ has maximum rank because $d(F_t)_x$ has maximum rank. Thus by the rank theorem (specifically the immersion version where the domain can have boundary) we get that for any $t_0 \in [0,1]$ and any $x_0 \in \partial M$ there exists a neighborhood I of t_0 in $[0,1]$ and a neighborhood V of x_0 in U_{r_0} such that G is injective over $I \times V$. Notice that this implies that for any $t \in I$, F_t is injective over V. Now, suppose that there does not exist an $r > 0$ such that F_t is injective over U_r for all $t \in [0,1]$. Then there exist sequences $\{t_k\}_{k=1}^{\infty}$, $\{x_k\}_{k=1}^{\infty} \subseteq U_{r_0}$, and $\{y_k\}_{k=1}^{\infty} \subseteq U_{r_0}$ such that each $x_k \neq y_k$, $F_{t_k}(x_k) = F_{t_k}(y_k)$, and $d_{\tilde{g}}(x_k, \partial M), d_{\tilde{g}}(y_k, \partial M) \to 0$. As before, since $[0,1]$ and N are compact we can assume that $t_k \to t_0$, $x_k \to x_0$, and $y_k \to y_0$ for some $t_0 \in [0,1]$ and $x_0, y_0 \in N$. As before we also have that $x_0, y_0 \in \partial M$. By the continuity of F and uniqueness of limits we also have that $F_{t_0}(x_0) = F_{t_0}(y_0)$. Since $\varphi|_{\partial M} = \mathrm{id}$ and thus $F_{t_0}(x_0) = x_0$ and $F_{t_0}(y_0) = y_0$, we in fact have that $x_0 = y_0$. Now, let I and V be as above with respect to t_0 and x_0. Then F_t is injective over V for all $t \in I$. Since t_k is eventually in I and x_k and y_k are eventually in V, this implies that that $F_{t_k}(x_k) = F_{t_k}(y_k)$ eventually. But that contradicts the assumption on the sequences $\{t_k\}_{k=1}^{\infty}$, $\{x_k\}_{k=1}^{\infty}$, and $\{y_k\}_{k=1}^{\infty}$. Thus indeed there must exist an $r > 0$ such that F_t is injective over U_r for all $t \in [0,1]$.

With the above we have proven that there exists an $r > 0$ such that for any $t \in [0,1]$, both dF_t is nonsingular and F_t is injective over U_r. Let $r > 0$ be any such number. This $r > 0$ will be the one we want if we can show that there exists some $s > 0$ such that for all $t \in [0,1]$, F_t maps U_r onto a neighborhood of U_s. Let's prove this by contradiction, suppose not! Since dF_t is a local diffeomorphism for all $t \in [0,1]$, we have that the image of U_r under F_t is open. Thus for the sake of contradiction we can conclude that there exist sequences $\{t_k\}_{k=1}^{\infty}$ and $\{y_k\}_{k=1}^{\infty} \subseteq U_{r_0}$ such that each $y_k \notin F_{t_k}[U_r]$ and $d_{\tilde{g}}(y_k, \partial M) \to 0$. As before, since $[0,1]$ and N are compact we can assume that $t_k \to t_0$ and $y_k \to y_0$ for some $t_0 \in [0,1]$ and $y_0 \in N$. As before we also have that $y_0 \in \partial M$. In addition, as before we have that $F(t_0, y_0) = y_0$ since $y_0 \in \partial M$.

Now, I claim that there exists a neighborhood J of t_0 in $[0,1]$ and a neighborhood W of y_0 in U_r such that $G\big[[0,1] \times U_r\big]$ contains the set $J \times W$ (observe that $G(t_0, y_0) = (t_0, y_0)$). If $t_0 \in (0,1)$, then this immediately follow from the inverse function theorem since $dG_{(t_0,y_0)}$ is nonsingular and hence G is an open map near (t_0, y_0). To see that this is also true in the endpoint cases as well, suppose that $t_0 = 0$ or $t_0 = 1$. Then take a smooth extension $\tilde{F} : \tilde{I} \times \tilde{V} \to N$ of F onto an open neighborhood $\tilde{I} \times \tilde{V}$ of (t_0, y_0) in $\mathbb{R} \times U_r$ (i.e. $\tilde{I}$ is an open interval in $\mathbb{R}$). Consider the map $\tilde{G} : \tilde{I} \times \tilde{V} \to \mathbb{R} \times N$ given by $\tilde{G}(t,x) = \left(t, \tilde{F}(t,x)\right)$. Then the inverse function implies that for some open neighborhood $\tilde{J}$ of t_0 in $\mathbb{R}$ and some open neighborhood W of y_0 in U_r, $\tilde{G}[\tilde{I} \times \tilde{V}]$ contains the set $\tilde{J} \times W$. Notice then that if we let $J = \tilde{J} \cap [0,1]$, then $G\big[[0,1] \times U_r\big]$ contains $J \times W$.

Thus, let J and W be as in the previous paragraph. Observe that this implies that for any $t \in J$, $F_t[U_r]$ contains the set W. Since t_k is eventually in J and y_k is eventually in W, we have that this implies that $y_k \in F_{t_k}[U_r]$ eventually. But that contradicts the assumption on the sequences $\{t_k\}_{k=1}^{\infty}$ and $\{y_k\}_{k=1}^{\infty}$. Hence indeed there must exist an $s > 0$ such that for all $t \in [0,1]$, F_t maps U_r onto a neighborhood of U_s. This proves the claim (finally!)

End of proof of claim

Onwards! Let $r, s > 0$ be as in the above claim. Observe then that for any $t \in [0,1]$ and any $y \in U_s$ there exists a unique $x \in U_r$ such that $\eta(t,x) = y$. Let $\chi : [0,1] \times U_s \to U_r$ denote the map that gives such an x for all such t and y:

$$\eta\big(t, \chi(t,y)\big) = y \qquad \forall t \in [0,1] \quad \forall y \in U_s.$$

Since $x \mapsto \eta_t(x)$ is a diffeomorphism over U_r for all fixed $t \in [0,1]$, the smoothness of χ follows from the implicit function theorem. Now, let $Y : [0,1] \times \overline{U}_{s/2} \to TN$ denote the smooth time-dependent vector field given by

$$Y_y = d\eta_{\chi(t,y)}\left(\frac{d}{dt}\bigg|_{(t,\chi(t,y))}\right).$$

We defined Y in the second argument over the closed set $\overline{U}_{s/2}$ for extension purposes below. Now, by an argument similar to the one given at the end of the previous claim (with χ instead of

η) it's not hard to see that there exists a $\rho > 0$ such that for any $t \in [0,1]$ the map $\chi_t : U_{s/2} \to N$ given by $\chi_t(y) = \chi(t,y)$ maps $U_{s/2}$ onto a neighborhood of U_ρ. The reason why such a neighborhood U_ρ of ∂M is great is that observe that for any $x \in U_\rho$, the curve $t \mapsto \eta_x(t)$ for $t \in [0,1]$ is an integral curve of Y since

$$\eta_x'(t) = d\eta_x\left(\frac{d}{dt}\bigg|_{(t,x)}\right) = d\eta_{\chi(t,\eta_x(t))}\left(\frac{d}{dt}\bigg|_{\left(t,\chi(t,\eta_x(t))\right)}\right) = Y_{\eta_x(t)}.$$

Note that this doesn't necessarily hold merely on $x \in U_r$ since in that case $\eta_x(t)$ could map outside of where Y is defined ($\overline{U}_{s/2}$ that is). Now, since $[0,1] \times \overline{U}_{s/2}$ is closed in $\mathbb{R} \times N$, we have that there exists a smooth extension $Z : \mathbb{R} \times N \to TN$ of $\tilde{Y}$ onto all of $\mathbb{R} \times N$. Let $F : \mathbb{R} \times \mathbb{R} \times N \to N$ denote the flow of Z where the convention on the arguments is that for any $(t,x) \in \mathbb{R} \times N$, the curve $s \mapsto F(s,t,x)$ is an integral curve of Z. The reason that F is defined over all of $\mathbb{R} \times \mathbb{R} \times N$ is that its complete since N is compact. Let $\psi : N \to N$ be the map given by $\psi(x) = F(1,0,x)$. By the properties of time dependent flows, we know that ψ is a diffeomorphism. We finish the proof of this theorem by proving that this is the map that we want in the statement of the theorem (up to restricting to M).

First let's prove that $\psi = \varphi$ on U_ρ. This isn't hard since for any $x \in U_\rho$ we have that $\eta_x : [0,1] \to N$ is an integral curve of Z satisfying $\eta_x(0) = x$ and thus

$$\varphi(x) = \eta_x(1) = F(1,0,x) = \psi(x).$$

Next let's prove that $M = \psi[M]$. We'll prove this by showing that $M \subseteq \psi[M]$ and that $M \supseteq \psi[M]$. First let's prove the first inclusion. Since $\varphi|_{\partial M} = \mathrm{id}$, we have that $\psi|_{\partial M} = \mathrm{id}$ as well. Since ψ is bijective, this implies that $M^{\mathrm{int}} \subseteq \psi[N \setminus \partial M]$. Since $N \setminus \partial M = M^{\mathrm{int}} \cup M^c$ and M^{int} and M^c are disjoint open subsets of N, we have that $\psi[N \setminus \partial M]$ is the disjoint union of the open subsets $\psi[M^{\mathrm{int}}]$ and $\psi[M^c]$ of N. Thus the connectivity of M^{int} implies that either $M^{\mathrm{int}} \subseteq \psi[M^{\mathrm{int}}]$ or $M^{\mathrm{int}} \subseteq \psi[M^c]$ (recall that the interiors of connected manifolds with boundary are also connected). Now, since $\psi = \varphi$ over $U_\rho \cap M^{\mathrm{int}}$ and φ by assumption maps points of $U_\rho \cap M^{\mathrm{int}}$ into M^{int}, we see that the first case must happen: $M^{\mathrm{int}} \subseteq \psi[M^{\mathrm{int}}]$. Thus indeed $M \subseteq \psi[M]$.

Now let's show inclusion in the other direction. As before, since $\psi|_{\partial M} = \mathrm{id}$ we have that $\psi[M^{\mathrm{int}}] \subseteq M^{\mathrm{int}} \cup M^c$. And since M^{int} is connected, we have that $\psi[M^{\mathrm{int}}]$ is also connected. The disjoint nature of M^{int} and M^c then imply that either $\psi[M^{\mathrm{int}}] \subseteq M^{\mathrm{int}}$ or $\psi[M^{\mathrm{int}}] \subseteq M^c$. At the end of the previous paragraph we showed that $\psi[M^{\mathrm{int}}] \cap M^{\mathrm{int}} \neq \emptyset$. Thus we must have that $\psi[M^{\mathrm{int}}] \subseteq M^{\mathrm{int}}$ and so $\psi[M] \subseteq M$.

Having proved inclusion in both directions we finally have that indeed $M = \psi[M]$. Thus we that the restriction $\psi|_M$ is a diffeomorphism from M to itself that is equal to φ over the open neighborhood $U_\rho \cap M$ of ∂M in M. Thus $\psi|_M$ is the map that we wanted to prove the existence of in the statement of the theorem.

■

5 Chapter 13

5.1 (Exercise 13.1.3) Two Versions of Non-Abelian X-Ray Transform

In this exercise they ask us to prove that the two versions of the Non-Abelian X-Ray Transform are related by

$$C_{\mathcal{A},-} = \left[C_{\mathcal{A},+}\right] \circ \alpha.$$

Adopt the notation from Definition 2.17 above. By the uniqueness of solutions to ODEs and the fact that φ is a flow (i.e. U_+ and U_- are cocycles: see Exercise 5.3.1 in the book), we have that for any suitable $t, s \in \mathbb{R}$,

$$U_-(x, v, t + s) = U_+(\varphi_t(x, v), s)U_-(x, v, t).$$

Plug in $t = \tau(x, v)$ and $s = -\tau(x, v)$ to get that

$$U_-(x, v, 0) = U_+\big(\varphi_t(x, v), -\tau(x, v)\big)U_-\big(x, v, \tau(x, v)\big),$$

$$\mathrm{id} = U_+(x, v)U_- \circ \alpha(x, v),$$

$$U_- \circ \alpha(x, v) = [U_+(x, v)]^{-1}.$$

Composing both sides with α, using the fact that $\alpha^2 = \mathrm{id}|_{\partial M}$, and plugging in the definitions of $C_{\mathcal{A},+}$ and $C_{\mathcal{A},-}$ gives us what we wanted:

$$C_{\mathcal{A},-} = \left[C_{\mathcal{A},+}\right] \circ \alpha.$$

5.2 Hodge Star and Vertical Derivative over Two-Manifolds

Suppose that (M, g) is a 2-dimensional Riemannian manifold and that A is an $n \times n$ matrix of smooth one-forms over M, which we'll think of as a function of the form $A : TM \to \mathbb{C}^{n\times n}$. Consider further the restriction $A : SM \to \mathbb{C}^{n\times n}$. Right before Lemma 13.4.1 the authors state that

$$\star A = -VA$$

where $\star$ is the Hodge star operator and V is the vertical derivative. Since the above formula holds entry wise, observe that this will follow if we can prove the same fact for any one-form ω over M:

$$\star\,\omega = -V\omega.$$

To do this, we do (the second equality here follows by ω's linearity in v when x is fixed)

$$V\omega(x,v) = \frac{d}{dt}\bigg|_{t=0} \omega(\cos(t)\,v + \sin(t)\,v^{\perp}) = \omega(-\sin(0)\,v + \cos(0)\,v^{\perp}) = \omega(v^{\perp}) = \langle \omega^{\sharp}, v^{\perp}\rangle$$

$$= \langle -\left[\left(\omega^{\sharp}\right)^{\perp}\right]^{\perp}, v^{\perp}\rangle = -\langle\left(\omega^{\sharp}\right)^{\perp}, v\rangle = -\left[\left(\omega^{\sharp}\right)^{\perp}\right]^{\flat}(v).$$

Hence we get that

$$V\omega = -\star\,\omega,$$

which is of course equivalent to what we wanted to show.

5.3 Eq 13.4.1 (in Lemma 13.4.1) Hodge Star of the Curvature Operator

Intro (you don't necessarily have to understand everything here): Suppose that (M,g) is a 2-dimensional Riemannian manifold and that A is an $n\times n$ matrix of smooth one-forms over M, which we'll think of as a function of the form $A : TM \to \mathbb{C}^{n\times n}$. Consider further the restriction $A : SM \to \mathbb{C}^{n\times n}$. Moreover, let us assume that A is Hermitian at every point of M and consider the connection $d_A = d + A$ of the trivial bundle $\mathbb{C}^n$ over M. Let $F_A = d_A \circ d_A$ denote the "curvature operator." In the middle of the proof of this lemma, the authors state the following relation:

$$\star\,F_A = X_{\perp}(A) - X(\star\,A) + [\star\,A, A],$$

which the authors prove follows from the following identity:

$$X_{\perp}A(x,v) - X(\star\,A)(x,v) = dA(v,v^{\perp}) \qquad \forall (x,v) \in SM.$$

The authors give a proof of this equation, but allow me to present an alternative proof. I don't consider my proof advantageous since I do it in coordinates, but I want to show it simply because it's sometimes nice to see two different proofs of one result.

Main content: Notice that since the above equation holds entry wise, it will be sufficient to prove the same equation for any smooth 1-form $\omega \in \Gamma(T^*M)$ over M:

$$X_{\perp}\omega(x,v) - X(\star\,\omega)(x,v) = d\omega(v,v^{\perp}) \qquad \forall (x,v) \in SM.$$

Choose any point $(x_0, v_0) \in SM$ where we want to show that this equation holds. Let $(U, (x^i))$ be coordinates of M and take the standard coordinates $(\pi_{TM}^{-1}[U], (x^i, v^i))$ of TM that they generate:

$$v^i \frac{\partial}{\partial x^i}\bigg|_x \longmapsto (x^1, \dots, x^n, v^1, \dots, v^n).$$

In this chart of TM, we have that the local coordinate expression of the function $\omega : TM \to \mathbb{C}$ generated by our 1-form $\omega = \omega_i dx^i$ is given by

$$\omega(x^1, \dots x^n, v^1, \dots, v^n) = \omega_i(x^1, \dots, x^n)v^i.$$

It's not hard to show that in coordinates,

$$X = v^k \frac{\partial}{\partial x^k} - \Gamma^k_{ij} v^i v^j \frac{\partial}{\partial v^k},$$

$$X_\perp = (v^\perp)^k \frac{\partial}{\partial x^k} - \Gamma^k_{ij} (v^\perp)^i v^j \frac{\partial}{\partial v^k},$$

$$\star\, \omega = \frac{1}{\det g} \begin{bmatrix} -\omega_2 & \omega_1 \end{bmatrix} \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix},$$

where equality in the last equation is interpreted as taking the first and second components of the resultant horizontal vector on the right (i.e. 1×2 matrix) and make them the components of dx^1 and dx^2 respectively.

Now, let's assume that our coordinates are in fact normal coordinates centered at x_0, so that all

$$g_{ij}\big|_{x_0} = \delta_{ij}, \qquad \partial_k g_{ij}\big|_{x_0} = 0, \qquad \Gamma^k_{ij}\big|_{x_0} = 0.$$

It's then not hard to show from the previous equations that

$$X_\perp \omega(x_0, v_0) = (v_0^\perp)^k \frac{\partial \omega_r}{\partial x^k} v_0^r,$$

$$X(\star\, \omega)(x_0, v_0) = v^k \frac{\partial \omega_r}{\partial x^k} (v_0^\perp)^r,$$

Subtracting these two equations (and relabeling $k \leftrightarrow r$ in the second equation) gives

$$X_\perp \omega(x_0, v_0) - X(\star\, \omega)(x_0, v_0) = \sum_{k,r=1}^{n} v^k \left(\frac{\partial \omega_r}{\partial x^k} - \frac{\partial \omega_k}{\partial x^r} \right) (v^\perp)^r = d\omega(v, v^\perp).$$

This is what we wanted to show.