# Haim's Notes About

## *Invariant Distributions, Beurling Transforms and Tensor Tomography in Higher Dimensions*

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# 1 Page 318 (PDF Page 14) Decomposition of X

**Decomposition of** $X$. The geodesic vector field behaves nicely with respect to the decomposition into fibrewise spherical harmonics: it maps $\Omega_m$ into $\Omega_{m-1} \oplus \Omega_{m+1}$ [23, Proposition 3.2]. Hence on $\Omega_m$ we can write

$$X = X_- + X_+$$

where $X_- : \Omega_m \to \Omega_{m-1}$ and $X_+ : \Omega_m \to \Omega_{m+1}$. By [23, Proposition 3.7] the operator $X_+$ is overdetermined elliptic (i.e. it has injective principal symbol). One can gain insight into why the decomposition $X = X_- + X_+$ holds as follows. Fix $x \in M$ and consider local coordinates which are geodesic at $x$ (i.e. all Christoffel symbols vanish at $x$). Then $Xu(x, v) = v^i \frac{\partial u}{\partial x^i}$. We now use the following basic fact about spherical harmonics: the product of a spherical harmonic of degree $m$ with a spherical harmonic of degree one decomposes as the sum of a spherical harmonics of degree $m-1$ and $m+1$. Since the $v^i$ have degree one, this explains why $X$ maps $\Omega_m$ to $\Omega_{m-1} \oplus \Omega_{m+1}$.

In this section I'd like to fill in the details on the highlighted items. First we prove the stated fact about products of spherical harmonics:

**Lemma 1.1:** *Suppose that $a \in H_1(S^{d-1})$ and $b \in H_m(S^{d-1})$ are spherical harmonics of order $1$ and $m$ respectively on the sphere $S^{d-1}$ sitting in Euclidean space $\mathbb{R}^d$ with respect to the (flat) Euclidean Laplacian. Then the product $ab$ is in $H_{m+1}(S^{d-1}) \oplus H_{m-1}(S^{d-1})$.*

**Proof:** Let $S = S^{d-1}$ and let $r^2$ denote the polynomial $(x_1)^2 + \cdots + (x_d)^2$ over $\mathbb{R}^d$. Let $\mathcal{P}_k$ denote the set of all homogeneous (complex valued) polynomials of degree $k$ in $\mathbb{R}^d$ and let

$$\mathcal{H}_k = \{P \in \mathcal{P}_k : \Delta P = 0\}.$$

Recall the standard fact that

$$H_k(S) = \{P|_S : P \in \mathcal{H}_k\}$$

and that these spaces are perpendicular to each other with respect to the $L^2(S)$ inner product (see for instance [1] and my notes about that book). Now, we have that $a = \tilde{a}|_S$ and $b = \tilde{b}\big|_S$ where $\tilde{a} \in \mathcal{H}_1$ and $\tilde{b} \in \mathcal{H}_m$. Hence $ab = \tilde{a}\tilde{b}\big|_S$ where clearly $\tilde{a}\tilde{b} \in \mathcal{P}_{m+1}$. Now, by Corollary 2.50 in [1] we have that $\tilde{a}\tilde{b} = \sum_{k=0}^{\lfloor m/2 \rfloor} f_{m+1-2k}$ where each $f_{m+1-2k} \in r^{2k}\mathcal{H}_{m+1-2k}$ and so $ab \in \bigoplus_{k=0}^{\lfloor m/2 \rfloor} H_{m+1-2k}(S)$. Hence the lemma will be proved if we can show that $ab \perp H_j(S)$ for $j \leq m-3$ with respect to the $L^2(S)$ inner product.

Fix any $j \leq m-3$ and take a basis $\{Y_\mu : \mu = 1, \dots, l\}$ of $H_k(S)$. We have to show that for any $\mu = 1, \dots, l$,

$$\int_S ab\overline{Y_\mu} = 0.$$

Observe that the integral on the left-hand side is equal to

$$\int_S b\overline{\overline{a}Y_\mu}.$$

Now, $\bar{a}$ and $Y_\mu$ are spherical harmonics of order 1 and $j$ respectively and hence by similar arguments as above their product is in $\bigoplus_{k=0}^{\lfloor j/2 \rfloor} H_{j+1-2k}(S)$. Since $b \in H_m(S)$ is perpendicular to the latter, we get that the above integral is indeed equal to zero.

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Next let's discuss why this implies that $X$ maps $\Omega_m$ into $\Omega_{m-1} \oplus \Omega_{m+1}$. Fix any integer $m \geq 0$ and take any $u \in \Omega_m$. Following the text, take any point $x_0$ and consider normal coordinates $(x^i)$ centered at $x_0$ which naturally generate the coordinates $v^j\, \partial/\partial x^j \mapsto (x^i, v^j)$ of $TM$. Let $(g_{ij})$ denote the metric tensor in these coordinates. Above the point $x_0$ we have that $X = v^i\, \partial/\partial x^i$. So the claim will follow from the above lemma if we show that the only possible nonzero Fourier mode of $\partial u_m/\partial x^i$ on the sphere above $x_0$ is $m$. Unfortunately, doing this in our coordinates of $TM$ is a little inconvenient, so we construct another set of coordinates.

Let $(b_i)$ be the smooth orthonormal frame over the domain of $(x^i)$ obtained by applying the Gram-Schmidt orthogonalization process to the frame $(\partial/\partial x^i)$. This frame gives us another set of coordinates of $TM$ given by $w^j b_j \mapsto (x^i, w^j)$. Let $(\alpha_\mu^\nu)$ be the coefficients in the relation $b_\mu = \alpha_\mu^\nu\, \partial/\partial x^\nu$. Thinking about how the Gram-Schmidt orthogonalization process works, it's not hard to see that each $\partial g_{ij}/\partial x^r$ being equal to zero at $x = x_0$ implies that all of the partials $\partial \alpha_\mu^\nu/\partial x^r$ are zero at $x = x_0$ as well (hint: use induction). Furthermore, if we let $(\beta_\mu^\nu)$ be the coefficients in the inverse relation $\partial/\partial x^\mu = \beta_\mu^\nu b_\nu$, it's not hard to see that the $\beta_\mu^\nu$'s share the same property of the $\alpha_\mu^\nu$'s mentioned in the previous sentence. From this observation we see that above $x_0$, $X = w^i\, \partial/\partial x^i$. So we simply need to show that on the sphere above $x_0$, the only possible nonzero Fourier mode of the partial $\partial u_m/\partial x^i$ taken with respect to $(x^i, w^j)$ is $m$.[1] We do this by showing that on the sphere above $x_0$, $\partial u_m/\partial x^i$ is perpendicular to Fourier modes of order other than $m$.

Choose some nonnegative integer $j \neq m$. Let $Y$ be a harmonic polynomial homogeneous of degree $j$ over $\mathbb{R}^n$ with respect to the (flat) Euclidean Laplacian. Consider the smooth function $\mathcal{Y}$ defined over $TM$ near $x_0$ given by

$$\mathcal{Y}\left(x, w^i b_i\right) = Y(w^1, \dots, w^n). \tag{1.2}$$

To prove our claim, it will be sufficient to show that $\langle \partial u_m/\partial x^i\,, \mathcal{Y}\rangle_{L^2(S_{x_0}M)} = 0$. Observe that the inner product on the left-hand side is equal to

$$\int_{S_{x_0}M} \frac{\partial u_m}{\partial x^i}(x_0, w)\mathcal{Y}(x_0, w)dw_{S_{x_0}M}$$

$$= \frac{\partial}{\partial x^i}\bigg|_{x=x_0}\left(\int_{S_{x_0}M} u_m(x,w)\mathcal{Y}(x,w)dw_{S_{x_0}M}\right) - \int_{S_{x_0}M} u_m(x_0,w)\frac{\partial}{\partial x^i}\bigg|_{x=x_0}\big(\mathcal{Y}(x,w)\big)dw_{S_{x_0}M}.$$

The first term on the right-hand side is equal to zero since $u_m$ is constantly perpendicular to the Fourier modes of order $j$. By (1.2) above, the second term is also equal to zero. Hence, we've proven the claim.

## 2 Page 349 (PDF Page 45) Differential of Distance Function

The hypersurface $SM$ in $TM$ is given by $SM = f^{-1}(1)$ where $f : TM \to \mathbb{R}$ is the function $f(x, y) = g_{jk}(x)y^j y^k$. A computation gives

$$df(X^j\delta_{x_j} + Y^k\partial_{y_k}) = 2y_kY^k.$$

In this section I'd like to fill in the details on the highlighted equation. If $f(x,y) = g_{\mu\nu}(x)y^\mu y^\nu$, then

$$df\left(X^j\delta_{x_j} + Y^k\partial_{y_k}\right) = X^j\left(\frac{\partial g_{\mu\nu}}{\partial x^j}y^\mu y^\nu - \Gamma_{jk}^l y^k 2g_{l\nu}y^\nu\right) + Y^k 2g_{k\nu}y^\nu.$$

Now, $g_{k\nu}y^\nu = y_k$ (i.e. we lower an index on $y$). Moreover, by renaming variables we can also rewrite the term

[1] This is a different task from before since, except at $x = x_0$, the partial $\partial u_m/\partial x^i$ is not necessarily the same thing with respect to the coordinates $(x^i, v^j)$ and $(x^i, w^j)$.

$$\Gamma^l_{jk} y^k 2 g_{l\nu} y^\nu = \Gamma^l_{j\mu} y^\mu g_{l\nu} y^\nu + \Gamma^l_{j\nu} y^\nu g_{l\mu} y^\mu.$$

Hence the right-hand side of the previous equation can be rewritten as

$$\left(\frac{\partial g_{\mu\nu}}{\partial x^j} - \Gamma^l_{j\mu} g_{l\nu} - \Gamma^l_{j\nu} g_{\mu l}\right) X^j y^\mu y^\nu + 2 y_k Y^k = \nabla g(X, y, y) + 2 y_k Y^k = 2 y_k Y^k$$

since $\nabla g \equiv 0$.

We can actually rewrite the quantity $y_k Y^k$ in a coordinate invariant way. To see how, first let's prove a lemma that's interesting in its own write. Take the projection map $\pi : TM \to M$ and recall the well-known connection map $K : TTM \to TM$, the latter of which is described in my notes about [2].

**Lemma 2.1:** *The sets $\{\delta_{x_j} : j = 1 \dots, n\}$ and $\{\partial_{y_k} : k = 1, \dots, n\}$ form bases for $ker\ K$ and $ker\ d\pi$ respectively. In particular, we get that $\{\delta_{x_j}, \partial_{y_k}\}$ form a basis of $T_{(x,y)}TM$ by the well-known that $TM = ker\ K \oplus ker\ d\pi$.*

**Proof:** It's clear that $\{\delta_{x_j}\}$ and $\{\partial_{y_k}\}$ are linearly independent sets of vectors. It's easy to see that each $\partial_{y_k} \in \ker d\pi$. The fact that $\delta_{x_j} \in \ker K$ follows from

$$K\left(\delta_{x_j}\right) = \left(\left(-\Gamma^k_{j\nu} y^\nu\right) + \Gamma^k_{j\nu} y^\nu\right) \partial_{x_k} = 0.$$

■

Since the $\delta_{x_j}$ are in the kernel of $K$ and $K$ maps $X^j \delta_{x_j} + Y^k \partial y_k$ to $Y^k \partial_{x^k}$, we see that the quantity $y_k Y^k$ can be rewritten in the coordinate invariant manner:

$$y_k Y^k = y^\flat K\left(X^j \delta_{x_j} + Y^k \partial y_k\right).$$

## 3 Page 350 (PDF page 46) Local Coordinate Expression for Decomposition of Gradient over *SM*

where $p : TM \backslash \{0\} \to SM$ is the projection $p(x, y) = (x, y/|y|_{g(x)})$. We see that the decomposition $\nabla_{SM} u = (Xu)X + \overset{\mathrm{h}}{\nabla} u + \overset{\mathrm{v}}{\nabla} u$ has the following form in local coordinates:

$$Xu = v^j \delta_j u,$$
$$\overset{\mathrm{h}}{\nabla} u = (\delta^j u - (v^k \delta_k u) v^j) \partial_{x_j},$$
$$\overset{\mathrm{v}}{\nabla} u = (\partial^k u) \partial_{x_k}.$$

In this section I'd like to show how these equations are derived. Let's start with the first one.

**Lemma 3.1:** *The following are true*

$$X = v^k \delta_{x_k} \quad in \quad TM,$$

$$X = v^k \delta_k \quad in \quad SM.$$

**Proof:** We have by the well-known equation for the geodesic vector field over $TM$ that (in the second equality below I change the index names)

$$X = v^k \partial_{x_k} - \Gamma_{ij}^k v^i v^j \partial_{y_k} = v^k \left( \partial_{x_k} - \Gamma_{kj}^l v^j \partial_{y_l} \right) = v^k \delta_{x_k}.$$

Now take any $u \in SM$. Observe that $Xu = X(u \circ p)$ since $X$ is tangent to $SM$. Hence

$$Xu = v^k \delta_{x_k}(u \circ p) = v^k \delta_k(u)$$

and so indeed $X = v^k \delta_k$ over $SM$.

■

Next let's derive the equation for $\overset{v}{\nabla} u$. Let $i_{SM} : SM \to TM$ denote the inclusion of $SM$ into $TM$. For any $u \in C^\infty(SM)$, we define $\delta^j(u)$ and $\partial^k(u)$ for $j, k = 1, \ldots, n$ to be the components

$$d i_{SM}(\operatorname{grad} u) = \delta^j(u) \delta_{x_j} + \partial^k(u) \partial_{y_k}.$$

Since $u \circ p(x, y)$ is unchanged when $y$ is scaled, it's not hard to see that $d i_{SM}(\operatorname{grad} u) = \operatorname{grad}(u \circ p)$ and hence the above equation can be rewritten as

$$\operatorname{grad}(u \circ p) = \delta^j(u) \delta_{x_j} + \partial^k(u) \partial_{y_k}.$$

As a side note, it's not hard to see that each operator $\delta^j$ and $\partial^k$ are linear and satisfy the property of a derivation and thus are tangent vectors to $SM$. Observe also that these two operators look like they are raising the indices of $u$. This is made precise by the following lemma.

**Lemma:** *The following are true:*

$$\delta^j = g^{ji} \delta_i,$$

$$\partial^k = g^{kr} \partial_r.$$

**Proof:** For any $u \in C^\infty(SM)$ and any $w \in TTM$ we have that (here $\langle \cdot, \cdot \rangle$ is the Sasaki metric – see my notes about [2]).

$$\langle \operatorname{grad}(u \circ p), w \rangle = \langle \delta^j(u) \delta_{x_j} + \partial^k(u) \partial_{y_k}, w^i \delta_{x_i} + w^r \partial_{y_r} \rangle = g_{ji} \delta^j(u) w^i + g_{kr} \partial^k(u) w^r.$$

On the other hand,

$$\langle \operatorname{grad}(u \circ p), w \rangle = w^i \delta_{x_i}(u \circ p) + w^r \partial_{y_r}(u \circ p).$$

Equating the two right-hand sides gives

$$g_{ij}\delta^j(u) = \delta_{x_i}(u \circ p),$$

$$g_{rk}\partial^k(u) = \partial_{y_r}(u \circ p).$$

From here the lemma follows.

■

Now, let $\pi : TM \to M$ denote the natural and recall the well-known connection map $K : TTM \to TM$, the latter of which is described in my notes about [2]. Let "$\mathrm{proj}_{\ker K} : TM \to \ker K$" and "$\mathrm{proj}_{\ker d\pi} : TM \to \ker d\pi$" denote the projection maps associated to the orthogonal decomposition $TM = \ker K \oplus \ker d\pi$. Then we have by definition that

$$\overset{\mathrm{v}}{\nabla} u = K\left(\mathrm{proj}_{\ker d\pi}\big(di_{SM}(\mathrm{grad}\, u)\big)\right) = K\big(\partial^k(u)\partial_{y_k}\big) = \partial^k(u)\partial_{x_k}.$$

Similarly we have that

$$\overset{\mathrm{h}}{\nabla} u = d\pi\big[\mathrm{proj}_{\ker K}\big(di_{SM}(\mathrm{grad}\, u)\big) - \langle di_{SM}(\mathrm{grad}\, u), X\rangle X\big].$$

Since $X$ is tangent to $SM$, it's not hard to see that the second quantity in the square brackets is $X(u)X$. Hence the above quantity is equal to

$$d\pi\left[\delta^j(u)\delta_{x_j}\right] - d\pi[X(u)X] = \delta^j(u)\partial_{x_j} - X(u)v^j\partial_{x_j}.$$

If we use Lemma 3.1 above, we can rewrite this last quantity as

$$\overset{\mathrm{h}}{\nabla} u = \big(\delta^j(u) - v^k\delta_k(u)v^j\big)\partial_{x_j}.$$

# 4 References

Additional works referenced above:


1. Folland, G. B. (1995). *Introduction to Partial Differential Equations* (2nd ed.). Princeton: Princeton University Press.
2. Paternain, G., Salo, M., & Uhlmann, G. (2022). *Geometric Inverse Problems, With Emphasis in Two Dimensions*. Cambridge: Cambridge University Press & Assessment.