Haim's Notes About
Introduction to Riemannian Manifolds (2nd ed)
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Notations and Conventions

Notation: For any integer $n \geq 1$, $\mathbb{H}^n$ denotes the upper-half space of $\mathbb{R}^n$:

$$\mathbb{H}^n = \{x \in \mathbb{R}^n : x^n \geq 0\}$$

Notation: The notation $\mathbb{R}_+$ denotes the set of positive real numbers:

$$\mathbb{R}_+ = \{x \in \mathbb{R} : x > 0\}.$$

Notation: The notation $\mathbb{Z}_+$ denotes the set of positive integers:

$$\mathbb{Z}_+ = \{1,2,3, \dots\}.$$

Convention: A neighborhood in a topology always mean an *open* neighborhood.

Convention: All appropriate structures are smooth (C^∞) unless stated otherwise. I'll often still include the word *smooth* for emphasis.

Definition: A **regular domain** is a smooth properly embedded submanifold with boundary of codimension zero.

Convention: I use the Einstein summation convention extensively here.

Convention: Suppose that (M, g) is a Riemannian manifold or pseudo-Riemannian manifold possibly with boundary. We let ∇ denote the total covariant derivative. Furthermore, in local coordinates we put indices arising from covariant differentiation after a semicolon. For instance, if F is a smooth covariant tensor field over M of rank 2, then in any local coordinates (x^i) we write

$$\nabla F = F_{ij;k} dx^i \otimes dx^j \otimes dx^k$$

where each

$$F_{ij;k} = \partial_k(F_{ij}) - \Gamma_{ki}^{\lambda} F_{\lambda j} - \Gamma_{kj}^{\lambda} F_{i\lambda}.$$

Indices of multiple covariant derivatives (e.g. ∇^3) are written from left to right after the semicolon (as one would naturally expect)

Convention: Suppose that (M, g) is a Riemannian manifold or pseudo-Riemannian manifold possibly with boundary. Our convention is that the **(1,3)-curvature tensor** field R is defined as following: for any smooth vector fields X, Y, Z and smooth covector field ω over M,

$$R(X, Y, Z, \omega) = \omega\left(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z\right).$$

where $[X, Y]$ denote the Lie bracket of X and Y. We write the components of R as ${R_{ijk}}^l$.

Convention: Suppose that (M, g) is a Riemannian manifold or pseudo-Riemannian manifold possibly with boundary. Our convention is that the **Riemann curvature tensor** field Rm is defined as the tensor obtained by lowering the last index of the (1,3)-curvature tensor. We write the components of R as R_{ijkl}

Convention: Suppose that (M, g) is a Riemannian manifold or pseudo-Riemannian manifold possibly with boundary. Our convention is that the **Ricci curvature tensor** field Rc is defined as the tensor obtained by taking the trace of first and last index of the (1,3)-curvature tensor. We write the components of Rc as R_{ij}.

Convention: Suppose that (M, g) is a Riemannian manifold or pseudo-Riemannian manifold possibly with boundary. Our convention is that the **scalar curvature** S is defined as the quantity obtained by taking the trace of first and second index of the Ricci curvature tensor.

Notation: Suppose that M is a smooth manifold possibly with boundary and that h and k are smooth covariant symmetric tensor fields over M of rank 2. Then the **Kulkarni-Nomizu product** of h and k is the covariant tensor field of rank 4 given by the following: for any smooth vector fields W, X, Y, Z over M,

$$h \overset{\text{KN}}{\wedge} k = h(W, Z)k(X, Y) + h(X, Y)k(W, Z) - h(W, Y)k(X, Z) - h(X, Z)k(W, Y).$$

The component version of this equation is of course given by:

$$\left(h \overset{\text{KN}}{\wedge} k\right)_{ijlm} = h_{im}k_{jl} + h_{jl}k_{im} - h_{il}k_{jm} - h_{jm}k_{il}.$$

First allow me to point out that the Kulkarni-Nomizu product satisfies the same exact symmetry/anti-symmetry properties of the Riemann curvature tensor in its indices. Secondly, the above is merely a special case of the more general and analogous definition of the Kulkarni-Nomizu product of symmetric covariant 2-tensors over any vector space. Lastly, I would use the more conventional symbol for the Kulkarni-Nomizu product, which is the wedge inside a circle. However, since I don't know how to insert that symbol into Microsoft Word, unless this document has been converted to LaTeX, I will use the above symbol instead.

Convention: Suppose that (M, g) is a Riemannian manifold or pseudo-Riemannian manifold possibly with boundary of dimension $n \geq 3$. Our convention is that the **Schouten tensor of $\boldsymbol{g}$** is the tensor given by

$$P = \frac{1}{n-2}\left(Rc - \frac{S}{2(n-1)} g\right).$$

Convention: Suppose that (M, g) is a Riemannian manifold or pseudo-Riemannian manifold possibly with boundary of dimension $n \geq 3$. Our convention is that the **Weyl tensor of $\boldsymbol{g}$** is the tensor given by

$$W = Rm - P \overset{\text{KN}}{\wedge} g$$

where $\overset{\text{KN}}{\wedge}$ is the Kulkarni-Nomizu product.

Notation: Suppose that (M, g) is a Riemannian manifold or pseudo-Riemannian manifold possibly with boundary and that F is a smooth covariant 2-tensor field over M. The **exterior derivative** of F is defined as the covariant 3-tensor field given by: for any smooth vector fields X, Y, Z over M,

$$(DF)(X, Y, Z) = -(\nabla F)(X, Y, Z) + (\nabla F)(X, Z, Y)$$

where ∇ denotes the total covariant derivative of course. In coordinates this equation becomes

$$(DF)_{ijk} = -F_{ij;k} + F_{ik;j}.$$

Convention: Suppose that (M, g) is a Riemannian manifold or pseudo-Riemannian manifold possibly with boundary of dimension $n \geq 3$. Our convention is that the Cotton tensor is the tensor given by

$$C = -DP$$

where D denotes the exterior derivative of course. In coordinates this equation becomes

$$C_{ijk} = P_{ij;k} - P_{ik;j}.$$

Chapter 6

Boundary Normal Coordinates (Example 6.44 [Page 183])

Here I work out the details of the construction of boundary normal coordinates (boundary coordinates that are also semi-geodesic coordinates). Recall that all structures here are considered smooth unless stated otherwise. First let's prove a fact about extending Riemannian metrics.

Lemma 1: *Suppose that g is a Riemannian metric on an open subset U of $\mathbb{H}^n$. Then there exists a neighborhood V of U open in $\mathbb{R}^n$ and a Riemannian metric $\tilde{g}$ on V such that $\tilde{g}$ extends g (i.e. the inclusion $i : U \to V$ is an isometry).*

Proof: Intuitively speaking, as a first step let's locally extend g to a neighborhood of any point $p \in U$ that's open $\mathbb{R}^n$. Precisely, take any point $p \in U$ and let's define a Riemannian metric $\tilde{g}_p$ on a neighborhood W_p of p that's open in $\mathbb{R}^n$ as follows:

Case $p \in U^{int}$ in $\mathbb{R}^n$'s topology: In this case let $W_p \subseteq U^{\text{int}}$ be a neighborhood of p and set $\tilde{g}_p = g$ on W_p.

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Case $p \in \partial U$ in $\mathbb{R}^n$'s topology: Let's first write g in its Euclidean components:

$$g = g_{ij}\, dx^i \otimes dx^j.$$

Now, take any index (i, j) such that $i \leq j$. Since the component g_{ij} is smooth, there exists a smooth function $\tilde{g}_{ij} : W_{ij} \to \mathbb{R}$ on a neighborhood W_{ij} of p open in $\mathbb{R}^n$ that agrees with g_{ij} on $W_{ij} \cap U$. Find such a $\tilde{g}_{ij}$ for all indices (i, j) such that $i \leq j$. For the other indices (i, j) where $i > j$, simply set $g_{ij} : W_{ij} \to \mathbb{R}$ to be equal to $g_{ji} : W_{ji} \to \mathbb{R}$. Then, if we define

$$W = \bigcap_{i,j \leq n} W_{ij}$$

we get a smooth covariant symmetric 2-tensor field h on W given by:

$$h = \tilde{g}_{ij}\, dx^i \otimes dx^j$$

that agrees with g on $W \cap U$. Now, h is positive definite at p since it's equal to g at that point. So by Sylvester's criterion and the continuity of the determinants of the principal minors of the matrix $[\tilde{g}_{ij}]$ we see that we can furthermore shrink W so that h is positive definite on W. After shrinking W in this manner, set $W_p = W$ and $\tilde{g}_p = h$.

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Great! Notice that in both cases $\tilde{g}_p$ are local extensions of g in the sense that the inclusion map $i : W_p \cap U \to W_p$ is an isometry. Now let's use these local extensions to construct the V and $\tilde{g}$ desired in the lemma. Let $V \subseteq \mathbb{R}^n$ be the following open neighborhood of U in $\mathbb{R}^n$:

$$V = \bigcup_{p \in U} W_p.$$

Let $\{\psi_p : V \to \mathbb{R}\}_{p \in U}$ be a smooth partition of unity over V subordinate to the open cover $\{W_p : p \in U\}$. Set $\tilde{g}$ to be the following Riemannian metric over V:

$$\tilde{g} = \sum_{p \in U} \psi_p \cdot \tilde{g}_p.$$

This is of course well defined since the ψ_p's are locally finite. And since each $\tilde{g}_p$ locally extends g and the ψ_p's add up to one, we get that $\tilde{g}$ extends g in the sense described in the lemma. Thus this is the V and $\tilde{g}$ that we wanted.

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Now let's prove a version of the above lemma on manifolds.

Lemma 2: *Suppose that $\widetilde{M}$ is a smooth manifold (without boundary) and that $M \subseteq \widetilde{M}$ is a regular domain in $\widetilde{M}$. Suppose also that M is endowed with a Riemannian metric g. Then there exists a Riemannian metric $\tilde{g}$ on $\widetilde{M}$ that extends g (i.e. the inclusion $i : M \to \widetilde{M}$ is an isometry).*

Proof: This is proved similarly to the previous lemma. Pick any point $p \in \widetilde{M}$. There are two cases that can happen here: $p \in M$ or $p \in M^c$. Suppose that the first case happens: $p \in M$. Then since M is an embedded submanifold with boundary in $\widetilde{M}$, there exists a chart (W_p, φ_p) of $\widetilde{M}$ such that either φ_p is an interior chart of M as well or $W_p \cap M$ is the half-slice $x^n \geq 0$ where (x^i) are the components of φ_p. Construct a Riemannian metric $\tilde{g}_p$ over W_p as follows:

Case φ_p is an interior chart of M: Simply set $\tilde{g}_p = g$ over W_p.

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Case $W_p \cap M$ is the half-slice $x^n \geq 0$: Let $\hat{\mathcal{U}}_p = \operatorname{range} \varphi_p$. Then we have that the restriction $\tilde{\varphi}_p : W_p \cap M \to \hat{\mathcal{U}}_p \cap \mathbb{H}^n$ is a smooth chart of M. Since g is smooth over M, we have that $\hat{g} = \tilde{\varphi}_p^{-1^*} g$ is smooth over $\hat{\mathcal{U}}_p \cap \mathbb{H}^n$. By the previous lemma we know that there exists a smooth extension $\hat{g}_E$ of $\hat{g}$ onto a neighborhood $\hat{\mathcal{V}} \subseteq \hat{\mathcal{U}}_p$ of $\hat{\mathcal{U}}_p \cap \mathbb{H}^n$ open in $\mathbb{R}^n$ (the condition $\hat{\mathcal{V}} \subseteq \hat{U}_p$ is a trivial modification to the previous lemma). Redefine W_p to instead be $\varphi^{-1}[\hat{\mathcal{V}}]$ and set $\tilde{g}_p = \varphi^* \hat{g}_E$ on W_p.

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Notice that in both cases $\tilde{g}_p$ are local extensions of g in the sense that the inclusion map $i : W_p \cap M \to W_p$ is an isometry.

Now suppose that the other case happens: $p \in M^c$. Since M is closed in $\widetilde{M}$ (since it's properly embedded) there exists a neighborhood W_p of p open in $\widetilde{M}$ that is disjoint from M. Let W_p be any such neighborhood and let $\tilde{g}_p$ be *any* Riemannian metric over W_p.

Great! Let $\{\psi_p : \widetilde{M} \to \mathbb{R}\}_{p\in\widetilde{M}}$ be a smooth partition of unity over $\widetilde{M}$ subordinate to the open cover $\{W_p : p \in \widetilde{M}\}$ of $\widetilde{M}$. Finally, let $\tilde{g}$ denote the following Riemannian metric over $\widetilde{M}$:

$$\tilde{g} = \sum_{p\in\widetilde{M}} \psi_p \cdot \tilde{g}_p.$$

Notice that by construction, for any $q \in M$ and any $p \in M^c$, $\psi_p \cdot \tilde{g}_p$ is equal to zero at q since $\operatorname{supp}\psi_p \subseteq W_p$ is disjoint from M. This combined with the facts that the $\tilde{g}_p$ for $p \in M$ locally extend g and that the ψ_p's add up to one shows that $\tilde{g}$ extends g in the sense described in the lemma. Thus this is the $\tilde{g}$ that we wanted.

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Now let's discuss the main topic of this section: the construction of boundary local coordinates. Suppose that (M, g) is a Riemannian manifold and take any point $p \in \partial M$ where we want to construct boundary normal coordinates for M. Let N denote the double of M constructed in such a way that $M \subseteq N$ is a regular domain in N. By the previous lemma we know that there exists a Riemannian metric $\tilde{g}$ on N that extends g in the sense that the inclusion $M \hookrightarrow N$ is an isometry. Let $E : X \subseteq \mathcal{E}_{\partial M} \to U$ denote the normal exponential map of ∂M in N that maps diffeomorphically onto a normal neighborhood U. Let $V : W \subseteq \partial M \to TN$ be a smooth vector field over a neighborhood $W \subseteq \partial M$ of p in ∂M such that V is unit-length, normal to ∂M, and points inside M. By shrinking if necessary, let's furthermore assume that W is the domain of a smooth chart (W, ψ) of ∂M. Then, E, V, and ψ together generate a Fermi chart $(\mathcal{O}, \phi)$ of N given by the equation:

$$\phi^{-1}(x^1, \dots, x^{n-1}, x^n) = E\left(x^n V_{\psi^{-1}(x^1,\dots,x^{n-1})}\right).$$

Great! With $(\mathcal{O}, \phi)$ in hand we are ready to construct the boundary normal coordinates for M in a neighborhood of p. First let's observe one thing. Let $\hat{\mathcal{U}} \subseteq \mathbb{R}^n$ denote the range of ϕ. Then:

Claim: ϕ maps $\mathcal{O} \cap M$ to $\hat{\mathcal{U}} \cap \{x^n \geq 0\}$ and $\mathcal{O} \cap M^c$ to $\hat{\mathcal{U}} \cap \{x^n < 0\}$.

Proof of Claim: Fix any point of the form $(x^1, \dots, x^{n-1}, 0) \in \hat{\mathcal{U}}$. Recall that by definition, the intersection of normal neighborhoods of embedded submanifolds with fibers of the normal bundle is starshaped with respect to zero. In this case this implies that the set of $t \in \mathbb{R}$ such that $(x^1, \dots, x^{n-1}, t) \in \hat{\mathcal{U}}$ is some interval (a, b) that contains zero. Consider then the smooth curve $\gamma : (a, b) \to N$ given by $t \mapsto \phi^{-1}(x^1, \dots, x^{n-1}, t)$. The claim will be proven if we show that $\gamma(t) \in M$ for $t \geq 0$ and $\gamma(t) \in M^c$ if $t < 0$. Let's first prove that $\gamma(t) \in M$ for $t \geq 0$. By definition of the normal exponential map, we have that $\gamma(0) \in \partial M \subseteq M$. Furthermore, by the injectivity of the exponential map we have that $\gamma(t) \notin \partial M$ for $t > 0$. So for $t > 0$, or in other words $t \in (0, b)$, $\gamma(t)$ lies in the two disjoint open set M^{int} and M^c in N. By the connectedness of the interval $(0, b)$ we then must have that $\gamma(t)$ lies exclusively in either M^{int} or M^c on this interval. But notice that since

$$\gamma'(0) = V_{\phi^{-1}(x^1,\dots,x^{n-1},0)}$$

and by construction this vector points inside M, we see that $\gamma(t)$ is guaranteed to lie in M on a sufficiently small interval of the form $[0, \varepsilon)$ where $\varepsilon > 0$. So we must indeed have that $\gamma(t) \in M$ for all $t \geq 0$.

The fact that $\gamma(t) \in M^c$ for $t < 0$ is proved similarly. Indeed: by the injectivity of the exponential map we have that $\gamma(t) \notin \partial M$ for $t < 0$. Since M^{int} and M^c are disjoint open set in N and $(a, 0)$ is connected, we have that $\gamma(t)$ must lie in either M^{int} or M^c for $t < 0$. But the above displayed equation implies that $\gamma(t)$ is guaranteed to lie in M^c on a sufficiently small interval of the form $(-\delta, 0)$ where $\delta > 0$. So indeed $\gamma(t) \in M^c$ for $t < 0$. This proves the claim.

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Thus $(\mathcal{O}, \phi)$ is a half-slice chart of M in N. So letting $\tilde{\mathcal{O}} = \mathcal{O} \cap M$ and $\tilde{\phi} = \phi|_{\mathcal{O}\cap M}$ be the restriction of this chart to $\mathcal{O} \cap M$, we get the chart $\left(\tilde{\mathcal{O}}, \tilde{\phi}\right)$ of M in a neighborhood of p. The coordinates of this chart are the boundary normal coordinates that we wanted to construct. Indeed, notice that the curves $t \mapsto \tilde{\phi}^{-1}(x^1, \dots, x^{n-1}, t)$ are unit-speed geodesics in M for $t \geq 0$ since they're the restriction of the geodesics $t \mapsto \phi^{-1}(x^1, \dots, x^{n-1}, t)$ in N to M. And by Gauss' Lemma for submanifolds we have that the curves $t \mapsto \phi^{-1}(x^{-1}, \dots, x^{n-1}, t)$ are constantly perpendicular to the level sets $x^n = t$ for $t > 0$ (they are also perpendicular to the level set $\{x^n = 0\} \subseteq \partial M$ since their velocity at $t = 0$ is equal to V).

Chapter 7

Contracted Bianchi Identities (Proposition 7.18 [Page 209])

Here I give my own proof of the following theorem since I don't understand the proof given in the book.

Theorem: *Suppose that (M, g) is a Riemannian n-manifold or pseudo-Riemannian manifold possibly with boundary. Then*

$$\operatorname{tr}_g(\nabla Rm) = -D(Rc),$$

$$\operatorname{tr}_g(\nabla Rc) = \frac{1}{2} dS.$$

where both traces are being taken in the first and last components. In components, these equations take the form

$$R_{\mu ijk;}{}^{\mu} = R_{ij;k} - R_{ik;j},$$

$$R_{\mu j;}{}^{\mu} = \frac{1}{2} S_{;j}.$$

Proof: Take any point $p \in M$. We will prove that the above equations hold at p. Let $\left(U, \left(x^i\right)\right)$ be normal coordinates of M centered at p (i.e. p is $(0, \ldots, 0)$ in these coordinates). Recall that in such coordinates the Christoffel symbols and the first partials of the entries of metric tensor and its inverse are zero at p: for any $i, j, k \in \{1, \ldots, n\}$,

$$\Gamma_{ij}^{k}\Big|_p = 0, \qquad \partial_k g_{ij}\Big|_p = 0, \qquad \partial_k g^{ij}\Big|_p = 0.$$

The reason for choosing normal coordinates is that it will make computing the components of the above traces at p *so much* simpler. We have by the differential Bianchi identity that for any $j, k, l \in \{1, \ldots, n\}$ (where not specified, all of the following quantities are being evaluated at p where possible)

$$\left[\mathrm{tr}_g(\nabla Rm)\right]_{jkl} = g^{\mu\nu} R_{\mu jkl;\nu} = g^{\mu\nu}\left(-R_{\mu j\nu k;l} - R_{\mu jl\nu;k}\right) = -g^{\mu\nu}\partial_l R_{\mu j\nu k} - g^{\mu\nu}\partial_k R_{\mu jl\nu}$$

$$= -\partial_l\left(g^{\mu\nu} R_{\mu j\nu k}\right) - \partial_k\left(g^{\mu\nu} R_{\mu jl\nu}\right) = -\partial_l\left(R_{\mu j}{}^{\mu}{}_{k}\right) - \partial_k\left(R_{\mu jl}{}^{\mu}\right) = \partial_l\left(R_{jk}\right) - \partial_k\left(R_{jl}\right)$$

$$= R_{jk;l} - R_{jl;k}.$$

This proves the first equation. To prove the second, we have that for any $j \in \{1, \ldots, n\}$

$$\left[\mathrm{tr}_g(\nabla Rc)\right]_j = g^{\mu\nu} R_{\mu j;\nu} = g^{\mu\nu}\partial_\nu\left(R_{\mu j}\right) = g^{\mu\nu}\partial_\nu\left(g^{rs} R_{r\mu js}\right) = g^{\mu\nu} g^{rs}\partial_\nu\left(R_{r\mu js}\right)$$

$$= g^{\mu\nu} g^{rs} R_{r\mu js;\nu} = g^{\mu\nu} g^{rs}\left(-R_{r\mu\nu j;s} - R_{r\mu s\nu;j}\right) = -g^{rs} g^{\mu\nu}\partial_s\left(R_{r\mu\nu j}\right) - g^{rs} g^{\mu\nu}\partial_j\left(R_{r\mu s\nu}\right)$$

$$= -g^{rs}\partial_s\left(g^{\mu\nu} R_{r\mu\nu j}\right) - g^{rs}\partial_j\left(g^{\mu\nu} R_{r\mu s\nu}\right) = -g^{rs}\partial_s\left(R_{r\mu}{}^{\mu}{}_{j}\right) - g^{rs}\partial_j\left(R_{r\mu s}{}^{\mu}\right)$$

$$= -g^{rs}\partial_s\left(R_{rj}\right) + g^{rs}\partial_j(R_{rs}) = -g^{rs} R_{rj;s} + \partial_j(g^{rs} R_{rs}) = -\left[\mathrm{tr}_g(\nabla Rc)\right]_j + \partial_j S.$$

Rearranging finally gives

$$\left[\mathrm{tr}_g(\nabla Rc)\right]_j = \frac{1}{2}\partial_j S$$

which is the second equation in the theorem.

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Conformal Transformation of the Cotton Tensor

Here I prove the following theorem.

Theorem: *Suppose that (M, g) is a Riemannian or pseudo-Riemannian manifold possibly with boundary of dimension $n \geq 3$. Suppose also that $f \in C^\infty(M)$ is a smooth function over M and consider the metric $\tilde{g} = e^{2f} g$. If we let C and W denote the Cotton and Weyl tensors of g respectively and $\tilde{C}$ denote the Cotton tensor of $\tilde{g}$, then the two Cotton tensors are related by*

$$\tilde{C} = C + grad\, f \lrcorner W$$

where $grad\ f \lrcorner W$ denotes the 3-tensor obtained by inserting $grad\ f$ into the first argument of W. In local coordinates, this equation takes the form

$$\tilde{C}_{ijk} = C_{ijk} + W^l{}_{ijk}\partial_l f.$$

Remark: Observe that the conformal invariance of the Cotton tensor in dimension 3 follows immediately from the above theorem since the Weyl tensor is always zero in dimension 3.

Proof: Let's agree on the convention here that if I put a tilde "$\tilde{}$" over something, then it's the quantity related to the metric $\tilde{g}$. Otherwise, it's the quantity related to g. For example, $\tilde{C}$ refers to the Cotton tensor of $\tilde{g}$ while C refers to the Cotton tensor of g. Furthermore, if I put a tilde "$\tilde{}$" over a semicolon in the index of a tensor, then the covariant derivative of that tensor is taken with respect to $\tilde{g}$. Otherwise, the covariant derivative is taken with respect to g.

The proof of this theorem really just boils down to a long calculation. We'll do some tricks along the way to minimize what must be written down, the first of which is choosing normal coordinates with respect to g. Take any point $p \in M$. We will prove that the equation in the theorem holds at this arbitrarily chosen point p. Let (x^i) denote normal coordinates for g centered at p. Recall that in such coordinates we have that the following two properties are satisfied:

1.) The Christoffel symbols of g vanish at p.

2.) The first partials of g vanish at p.

Now, for any fixed $i, j, k \in \{1, \dots, n\}$ we have that

$$\tilde{C}_{ijk} = \tilde{P}_{ij\tilde{;}k} - \tilde{P}_{ik\tilde{;}j}$$

(note the tildes, even on the semicolons). Let's take a look at what the expression for the first quantity on the right-hand side $\tilde{P}_{ij\tilde{;}k}$ is equal to at p. The expression for the other quantity $\tilde{P}_{ij\tilde{;}k}$ evaluated at p will be exactly the same but with all of the j's and k's interchanged. From now on *every quantity I write is being evaluated at p*, even though I won't explicitly write it so. We have that

$$\tilde{P}_{ij\tilde{;}k} = \partial_k \tilde{P}_{ij} - \tilde{\Gamma}^{\lambda}_{ki}\tilde{P}_{\lambda j} - \tilde{\Gamma}^{\lambda}_{kj}\tilde{P}_{i\lambda}.$$

Plugging in the results of Proposition 7.29 and Theorem 7.30 (conformal transformation of the Christoffel symbols and curvature tensors) in the book gives us that (here I use property (1) from above)

$$\tilde{P}_{ij\tilde{;}k} = \partial_k\left(P_{ij} - \partial_{ij}f + \Gamma^{\lambda}_{ij}\partial_\lambda f + \partial_i f\partial_j f - \frac{1}{2}g^{\mu\nu}\partial_\mu f\partial_\nu f g_{ij}\right)$$

$$-\left(\partial_k f\delta^{\lambda}_{i} + \partial_i f\delta^{\lambda}_{k} - g^{\lambda r}\partial_r f g_{ki}\right)\left(P_{\lambda j} - \partial_{\lambda j}f + \partial_\lambda f\partial_j f - \frac{1}{2}g^{\mu\nu}\partial_\mu f\partial_\nu f g_{\lambda j}\right) - \tilde{\Gamma}^{\lambda}_{kj}\tilde{P}_{i\lambda}.$$

Before we continuity let me make two remarks. First of all, I didn't simply erase Γ_{ij}^{λ} inside the above ∂_k partial since it is not true that the partials of the Christoffel symbols are zero at p. Second, I didn't plug in the results of the two theorems mentioned into the last quantity $\tilde{\Gamma}_{kj}^{\lambda}\tilde{P}_{i\lambda}$ of a reason that will be clear later. Distributing the above ∂_k partial and the $(\ldots)(\ldots)$ quantity, and then using properties (1) and (2) above to simplify gives us that

$$\tilde{P}_{ij\,\tilde{;}\,k} = \partial_k P_{ij} - \partial_{kij} f + \partial_k \Gamma_{ij}^{\lambda} \partial_\lambda f + \partial_{ki} f \partial_j f + \partial_i f \partial_{kj} f - g^{\mu\nu} \partial_{k\mu} f \partial_\nu f g_{ij}$$

$$-\partial_k f \cdot P_{ij} + \partial_k f \partial_{ij} f - \partial_k f \partial_i f \partial_j f + \frac{1}{2} \partial_k f g^{\mu\nu} \partial_\mu f \partial_\nu f g_{ij}$$

$$-\partial_i f \cdot P_{kj} + \partial_i f \partial_{kj} f - \partial_i f \partial_k f \partial_j f + \frac{1}{2} \partial_i f g^{\mu\nu} \partial_\mu f \partial_\nu f g_{kj}$$

$$+g^{\lambda r} \partial_r f g_{ki} P_{\lambda j} - g^{\lambda r} \partial_r f g_{ki} \partial_{\lambda j} f + g^{\lambda r} \partial_r f g_{ki} \partial_\lambda f \partial_j f - \frac{1}{2} g^{\lambda r} \partial_r f g_{ki} g^{\mu\nu} \partial_\mu f \partial_\nu f g_{\lambda j} - \tilde{\Gamma}_{kj}^{\lambda} \tilde{P}_{i\lambda}.$$

Now, as mentioned before, the expression for $\tilde{P}_{ik\,\tilde{;}\,j}$ is exactly the same except that all of the j's and k's are interchanged. Imagine that I write out that other expression too and subtract it from the above expression for $\tilde{P}_{ij\,\tilde{;}\,k}$ to get an expression for $\tilde{P}_{ij\,\tilde{;}\,k} - \tilde{P}_{ik\,\tilde{;}\,j}$ (this requires some imagination). Almost all of the terms will cancel. To see this, let's discuss what terms in the above expression for $\tilde{P}_{ij\,\tilde{;}\,k}$ would cancel after we do such a subtraction. Observe that in the above expression for $\tilde{P}_{ij\,\tilde{;}\,k}$ the following terms are symmetric in j and k (colors appear here to help locate the terms in the above expression):

$$\partial_{kij} f, \quad \partial_i f \partial_{kj} f, \quad \partial_k f \partial_i f \partial_j f, \quad \partial_i f \cdot P_{kj}, \quad \partial_i f \partial_{kj} f, \quad \partial_i f \partial_k f \partial_j f,$$

$$\frac{1}{2} \partial_i f g^{\mu\nu} \partial_\mu f \partial_\nu f g_{kj}, \quad \tilde{\Gamma}_{kj}^{\lambda} \tilde{P}_{i\lambda},$$

$$\partial_{ki} f \partial_j f + \partial_k f \partial_{ij} f, \quad g^{\mu\nu} \partial_{k\mu} f \partial_\nu f g_{ij} + g^{\lambda r} \partial_r f g_{ki} \partial_{\lambda j} f,$$

and one more that requires some algebraic simplification:

$$\frac{1}{2} \partial_k f g^{\mu\nu} \partial_\mu f \partial_\nu f g_{ij} + g^{\lambda r} \partial_r f g_{ki} \partial_\lambda f \partial_j f - \frac{1}{2} g^{\lambda r} \partial_r f g_{ki} g^{\mu\nu} \partial_\mu f \partial_\nu f g_{\lambda j}$$

$$= \frac{1}{2} \partial_k f g^{\mu\nu} \partial_\mu f \partial_\nu f g_{ij} + g^{\lambda r} \partial_r f g_{ki} \partial_\lambda f \partial_j f - \frac{1}{2} \partial_j f g_{ki} g^{\mu\nu} \partial_\mu f \partial_\nu f$$

$$= \frac{1}{2} \partial_k f g^{\mu\nu} \partial_\mu f \partial_\nu f g_{ij} + \frac{1}{2} \partial_j f g^{\mu\nu} \partial_\mu f \partial_\nu f g_{ki}.$$

Thus these terms will also appear in the expression for $\tilde{P}_{ik\,\tilde{;}\,j}$ and hence will cancel out when we subtract $\tilde{P}_{ik\,\tilde{;}\,j}$ from $\tilde{P}_{ij\,\tilde{;}\,k}$. Therefore, after all such cancellation we're only left with (colors here will again be used to help locate terms in what appears afterwards):

$$\tilde{C}_{ijk} = \tilde{P}_{ij\,\tilde{;}\,k} - \tilde{P}_{ik\,\tilde{;}\,j} = \partial_k P_{ij} + \partial_k \Gamma_{ij}^{\lambda} \partial_\lambda f - \partial_k f \cdot P_{ij} + g^{\lambda r} \partial_r f g_{ki} P_{\lambda j}$$

$$-\partial_j P_{ik} - \partial_j \Gamma_{ik}^{\lambda} \partial_\lambda f + \partial_j f \cdot P_{ik} - g^{\lambda r} \partial_r f \, g_{ji} P_{\lambda k}$$

Now by property (1) above we have that $\partial_k P_{ij}$ and $\partial_j P_{ik}$ are equal to $P_{ij\,;\,k}$ and $P_{ik\,;\,j}$ respectively. By property (1) again we have that

$$\partial_k \Gamma_{ij}^{\lambda} \partial_\lambda f - \partial_j \Gamma_{ik}^{\lambda} \partial_\lambda f = g_{\lambda l}\left(\partial_k \Gamma_{ij}^{\lambda} - \partial_j \Gamma_{ik}^{\lambda}\right)(\operatorname{grad} f)^l = R_{kjil}(\operatorname{grad} f)^l = R_{lijk}(\operatorname{grad} f)^l$$

where in the last equality I used the symmetry/anti-symmetry properties of the indices of the Rm. For the remaining four terms we have that

$$-\partial_k f \cdot P_{ij} + g^{\lambda r} \partial_r f \, g_{ki} P_{\lambda j} + \partial_j f \cdot P_{ik} - g^{\lambda r} \partial_r f \, g_{ji} P_{\lambda k}$$

$$= -g_{k\lambda} P_{ij} (\operatorname{grad} f)^{\lambda} + g_{ki} P_{\lambda j} (\operatorname{grad} f)^{\lambda} + g_{j\lambda} P_{ik} (\operatorname{grad} f)^{\lambda} - g_{ji} P_{\lambda k} (\operatorname{grad} f)^{\lambda}$$

$$= \left(P \overset{\text{KN}}{\wedge} g\right)_{jki\lambda} (\operatorname{grad} f)^{\lambda}$$

$$= -\left(P \overset{\text{KN}}{\wedge} g\right)_{\lambda ijk} (\operatorname{grad} f)^{\lambda}$$

Thus applying all of the above simplifications gives us that

$$\tilde{C}_{ijk} = P_{ij\,;\,k} - P_{ik\,;\,j} + R_{lijk}(\operatorname{grad} f)^l - \left(P \overset{\text{KN}}{\wedge} g\right)_{lijk} (\operatorname{grad} f)^l$$

$$= C_{ijk} + W_{lijk}(\operatorname{grad} f)^l = C_{ijk} + W^{l}{}_{ijk} \partial_l f.$$

This proves the theorem.

■

The Weyl-Schouten Theorem

I don't fully understand the proof in the book for this theorem, and so here I give my own presentation of it. It's probably the same thing with the only difference being that I do the calculation needed for the Frobenius theorem a bit more explicitly.

Theorem: *Suppose that (M, g) is a Riemannian or pseudo-Riemannian manifold possibly with boundary of dimension $n \geq 3$.*

1.) If $n = 3$, then (M, g) is locally conformally flat if and only if its Cotton tensor is identically equal to zero.

2.) If $n \geq 4$, then (M, g) is locally conformally flat if and only if its Weyl tensor is identically equal to zero.

Proof: First suppose that (M, g) is locally conformally flat. If $n = 3$, we have that the Cotton tensor is identically zero since the Cotton tensor is conformally invariant in dimension three (see "Conformal Transformation of the Cotton Tensor") and the Cotton tensor of Euclidean space is trivially zero. If $n \geq 4$, then the Weyl tensor is zero because the Weyl tensor being zero is a conformally invariant phenomenon and the Weyl tensor of Euclidean space is also trivially zero (see Corollary 7.31 in the book).

Now suppose that the Cotton tensor is identically zero if $n = 3$ or that the Weyl tensor is identically zero if $n \geq 4$. I claim that in both cases, *both* the Cotton tensor and the Weyl tensor are identically zero. If $n = 3$, the fact that the Weyl tensor is zero follows from the well-known fact that Weyl tensor is always equal to zero in dimension 3 (see Corollary 7.26 in the book). If $n \geq 4$, then the fact that the Cotton tensor is zero follows from the well-known equation

$$\operatorname{tr}_g \nabla W = (n-3)C,$$

where the trace on the left-hand side is being taken in the first and last index (see Proposition 7.32 in the book). We will now show that g is locally conformally flat.

Take any point $p \in M$. We will show that there exists a smooth function $f \in C^\infty(U)$ over some (open) neighborhood U of p such that if we consider the metric $\tilde{g} = e^{2f} g$, then the Schouten tensor $\tilde{P}$ associated to $\tilde{g}$ will be zero. I claim that this will then imply that $\tilde{g}$ is flat and hence that g is conformally flat over U. To see this, let Rm, P, W and $\widetilde{Rm}, \tilde{P}, \widetilde{W}$ denote the Riemann curvature, Schouten, and Weyl tensors respectively for g and $\tilde{g}$ respectively. Since $W = 0$ we have that $\widetilde{W} = 0$ because, as mentioned before, the Weyl tensor being zero is a conformally invariant phenomenon. Thus, we have that

$$\widetilde{Rm} = \tilde{P} \overset{\text{KN}}{\wedge} \tilde{g}.$$

From this equation we indeed see that if we find an f such that $\tilde{P} \equiv 0$, then we'll have that the Riemann curvature tensor of $\tilde{g}$ is identically zero and hence that $\tilde{g}$ is flat. Thus, the proof will be finished if we find such an f.

By the result of Proposition 7.30 (conformal transformation of the curvature tensors), we have that we need to find an f that satisfies the following over U:

$$\tilde{P} = P - \nabla(df) + df \otimes df - \frac{1}{2}\langle df, df\rangle_g g \equiv 0.$$

Our approach will be to first find a covector field ξ over U that satisfies

$$P - \nabla\xi + \xi \otimes \xi - \frac{1}{2}\langle \xi, \xi\rangle_g g \equiv 0$$

and then find f that satisfies $df = \xi$. Ok, first let's prove the existence of a solution to the above equation. Suppose that U is the domain of some coordinates (x^i) of M. Notice that in such coordinates the above equation can be written as the following overdetermined system of first order partial differential equations:

$$\frac{\partial \xi_k}{\partial x^i} = \Gamma_{ki}^{\lambda}\xi_\lambda + P_{ki} + \xi_k\xi_i - \frac{1}{2}g^{\mu\nu}\xi_\mu\xi_\nu g_{ki} \qquad \forall k, i \in \{1, \dots, n\}.$$

For every $k, i \in \{1, \dots, n\}$, let $\alpha_i^k : U \times \mathbb{R}^n$ be the function given by

$$\alpha_i^k(x^1,\dots,x^n,z_1,\dots,z_n) = \Gamma_{ki}^{\lambda} z_\lambda + P_{ki} + z_k z_i - \frac{1}{2} g^{\mu\nu} z_\mu z_\nu g_{ki}.$$

Thus our overdetermined system can be rewritten as $\partial \xi_k / \partial x^i = \alpha(x^1,\dots,x^n,\xi_1,\dots,\xi_n)$. A theorem in the theory of overdetermined systems which is proved using the Frobenius theorem [Proposition 19.29 in John M. Lee's *Introduction to Smooth Manifolds 2nd Ed*] tells us that a solution to this overdetermined system exists in some neighborhood of p in U if and only if the compatibility conditions

(Eq 1)
$$\frac{\partial \alpha_i^k}{\partial x^j} + \alpha_j^r \frac{\partial \alpha_i^k}{\partial z^r} = \frac{\partial \alpha_j^k}{\partial x^i} + \alpha_i^r \frac{\partial \alpha_j^k}{\partial z^r}.$$

are satisfied (the necessity here is clear because the above merely states that $\partial^2 \xi_k / \partial x^i \partial x^j = \partial^2 \xi_k / \partial x^j \partial x^i$). To do this, we will instead show that the left-hand side of the above equation minus the right-hand side is equal to zero. First let's compute the expression for the left-hand side. The expression for the right-hand side will be exactly the same except that all of the i's and j's will be switched. We have that the left-hand side is equal to

$$\frac{\partial \alpha_i^k}{\partial x^j} + \alpha_j^r \frac{\partial \alpha_i^k}{\partial z^r} = \partial_j \Gamma_{ki}^{\lambda} z_\lambda + \partial_j P_{ki} - \frac{1}{2} \partial_j g^{\mu\nu} z_\mu z_\nu g_{ki} - \frac{1}{2} g^{\mu\nu} z_\mu z_\nu \partial_j g_{ki}$$
$$+ \left(\Gamma_{rj}^{\lambda} z_\lambda + P_{rj} + z_r z_j - \frac{1}{2} g^{\mu\nu} z_\mu z_\nu g_{rj} \right) (\Gamma_{ki}^r + \delta_k^r z_i + z_k \delta_i^r - g^{rs} z_s g_{ki})$$

Distributing the $(\dots)(\dots)$ quantity gives (colors appear here to help locate the terms later)

$$\frac{\partial \alpha_i^k}{\partial x^j} + \alpha_j^r \frac{\partial \alpha_i^k}{\partial z^r} = \partial_j \Gamma_{ki}^{\lambda} z_\lambda + \partial_j P_{ki} - \frac{1}{2} \partial_j g^{\mu\nu} z_\mu z_\nu g_{ki} - \frac{1}{2} g^{\mu\nu} z_\mu z_\nu \partial_j g_{ki}$$
$$+ \Gamma_{rj}^{\lambda} z_\lambda \Gamma_{ki}^r + \Gamma_{kj}^{\lambda} z_\lambda z_i + \Gamma_{ij}^{\lambda} z_\lambda z_k - \Gamma_{rj}^{\lambda} z_\lambda g^{rs} z_s g_{ki}$$
$$+ P_{rj} \Gamma_{ki}^r + P_{kj} z_i + P_{ij} z_k - P_{rj} g^{rs} z_s g_{ki}$$
$$+ z_r z_j \Gamma_{ki}^r + z_k z_j z_i + z_i z_j z_k - z_r z_j g^{rs} z_s g_{ki}$$
$$- \frac{1}{2} g^{\mu\nu} z_\mu z_\nu g_{rj} \Gamma_{ki}^r - \frac{1}{2} g^{\mu\nu} z_\mu z_\nu g_{kj} z_i - \frac{1}{2} g^{\mu\nu} z_\mu z_\nu g_{ij} z_k + \frac{1}{2} g^{\mu\nu} z_\mu z_\nu g_{rj} g^{rs} z_s g_{ki}.$$

Observe that the following term in the above expression is in fact equal to zero (here I use the fact that the covariant derivative of g is zero):

$$\frac{1}{2} \partial_j g^{\mu\nu} z_\mu z_\nu g_{ki} + \Gamma_{rj}^{\lambda} z_\lambda g^{rs} z_s g_{ki} = -\Gamma_{jr}^{\mu} g^{r\nu} z_\mu z_\nu g_{ki} + \Gamma_{rj}^{\lambda} z_\lambda g^{rs} z_s g_{ki} = 0.$$

Now, as mentioned before, the expression for the right-hand side of (Eq 1) is exactly the same except that all of the i's and j's are interchanged. Imagine that I write out that other expression too and subtract it from the above expression to get an expression for the left-hand side of (Eq 1) minus right-hand side of (Eq 1) (this requires some imagination). Almost all of the terms will cancel. To see this, let's discuss what terms in the above expression would cancel after we do such a subtraction. Observe that in the above expression the following terms are symmetric in i and j:

$$\Gamma_{ij}^{\lambda} z_\lambda z_k, \qquad P_{ij} z_k, \qquad z_k z_j z_i, \qquad z_i z_j z_k, \qquad \frac{1}{2} g^{\mu\nu} z_\mu z_\nu g_{ij} z_k,$$

$$\Gamma_{kj}^{\lambda} z_\lambda z_i + z_r z_j \Gamma_{ki}^{r},$$

and two more that require some algebraic simplification:

$$\frac{1}{2} g^{\mu\nu} z_\mu z_\nu \partial_j g_{ki} + \frac{1}{2} g^{\mu\nu} z_\mu z_\nu g_{rj} \Gamma_{ki}^{r} = \frac{1}{2} g^{\mu\nu} z_\mu z_\nu g_{kr} \Gamma_{ij}^{r} + \frac{1}{2} g^{\mu\nu} z_\mu z_\nu g_{ri} \Gamma_{kj}^{r} + \frac{1}{2} g^{\mu\nu} z_\mu z_\nu g_{rj} \Gamma_{ki}^{r},$$

and

$$z_r z_j g^{rs} z_s g_{ki} + \frac{1}{2} g^{\mu\nu} z_\mu z_\nu g_{kj} z_i - \frac{1}{2} g^{\mu\nu} z_\mu z_\nu g_{rj} g^{rs} z_s g_{ki}$$

$$= z_r z_j g^{rs} z_s g_{ki} + \frac{1}{2} g^{\mu\nu} z_\mu z_\nu g_{kj} z_i - \frac{1}{2} g^{\mu\nu} z_\mu z_\nu z_j g_{ki}$$

$$= \frac{1}{2} g^{\mu\nu} z_\mu z_\nu z_j g_{ki} + \frac{1}{2} g^{\mu\nu} z_\mu z_\nu z_i g_{kj}.$$

Thus these terms will also appear in the expression for the right-hand side of (Eq 1) and hence will cancel out when we subtract the right-hand side of (Eq 1) from the left-hand side of (Eq 1). Therefore, after all such cancellation we're only left with (here colors appear to help indicate what turns into what):

$$\left(\frac{\partial \alpha_i^k}{\partial x^j} + \alpha_j^r \frac{\partial \alpha_i^k}{\partial z^r}\right) - \left(\frac{\partial \alpha_j^k}{\partial x^i} + \alpha_i^r \frac{\partial \alpha_j^k}{\partial z^r}\right)$$

$$= \left(\partial_j \Gamma_{ki}^{\lambda} z_\lambda + \partial_j P_{ki} + \Gamma_{rj}^{\lambda} z_\lambda \Gamma_{ki}^{r} + P_{rj} \Gamma_{ki}^{r} + P_{kj} z_i - P_{rj} g^{rs} z_s g_{ki}\right)$$

$$-\left(\partial_i \Gamma_{kj}^{\lambda} z_\lambda + \partial_i P_{kj} + \Gamma_{ri}^{\lambda} z_\lambda \Gamma_{kj}^{r} + P_{ri} \Gamma_{kj}^{r} + P_{ki} z_j - P_{ri} g^{rs} z_s g_{kj}\right)$$

$$= R_{jikr} z^r + C_{kij} - \left(P \overset{\text{KN}}{\wedge} g\right)_{jikr} z^r$$

$$= W_{jikr} z^r + C_{kij},$$

where z^r denotes $g^{r\lambda} z_\lambda$ (I technically have to mention this since the z_λ's are not tensors but variables, and hence you can't technically raise their index). As we proved earlier, both the Weyl and Cotton tensors are zero and hence the above quantity is zero. Thus equality in (Eq 1) holds, and so we have that a solution to the mentioned overdetermined system for ξ indeed exists in some neighborhood of p. Let ξ be any such solution over some neighborhood $V \subseteq U$ of p.

For our last step, we need do show that there exists an f such that $df = \xi$ in some neighborhood of p. By the Poincaré Lemma we have that this will be true if we can show that ξ satisfies

$$\frac{\partial \xi_k}{\partial x^i} = \frac{\partial \xi_i}{\partial x^k}.$$

Looking back at our overdetermined system, we see that this is equivalent to showing that $\alpha_i^k = \alpha_k^i$. But this is immediately seen from the fact that the expression for α_i^k is symmetric in i and k. Thus the above condition holds and hence the theorem is proved.

■