Haim's Notes About

Introduction to Smooth Manifolds (2nd ed)

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Notations and Conventions

Notation: For any integer $n \geq 1$, $\mathbb{H}^n$ denotes the upper-half space of $\mathbb{R}^n$:

$$\mathbb{H}^n = \{x \in \mathbb{R}^n : x^n \geq 0\}$$

Notation: The notation $\mathbb{R}_+$ denotes the set of positive real numbers:

$$\mathbb{R}_+ = \{x \in \mathbb{R} : x > 0\}.$$

Notation: The notation $\mathbb{Z}_+$ denotes the set of positive integers:

$$\mathbb{Z}_+ = \{1,2,3, \dots\}.$$

Convention: A neighborhood in a topology always mean an *open* neighborhood.

Chapter 9

Boundary Flowout Theorem (Page 222 Theorem 9.24 or Problem 9.11)

Here I work out the proof of the following theorem that appears in this book.

Boundary Flowout Theorem: *Suppose that M is a smooth manifold with nonempty boundary and that N is a smooth vector field over M such that N is inward-pointing at any point of the boundary ∂M. Then there exists a smooth positive function $\delta : \partial M \to \mathbb{R}_+$ and a smooth map $\Phi : \mathcal{P}_\delta \to M$, where*

$$\mathcal{P}_\delta = \{(t,p) : p \in \partial M \quad \textit{and} \quad 0 \leq t < \delta(p)\},$$

such that Φ is a smooth embedding onto an open neighborhood of ∂M that satisfies the following property: for any $p \in \partial M$, the curve $t \mapsto \Phi(t,p)$ for $0 \leq t < \delta(p)$ is the integral curve of N starting at p.

The following proof follows the first half of the hint given in the book for this problem. The plan of attack to prove this theorem is the following:

1.) We can't use flow theory directly on the manifold because in the book we haven't yet constructed a theory of flows for manifolds with boundary. So what we will do is use boundary charts to push N down to local coordinates, extend N smoothly to an open set of the Euclidean space in those coordinates, and then use our ordinary "without boundary" flow theory on that to construct our Φ locally.

2.) After defining Φ locally, we will glue those local constructions together using a smooth partition of unity to get the global Φ desired in the theorem.

3.) Then we prove that the Φ that we construct satisfies all of the properties stated in the theorem.

Before we prove the theorem though, we need to generalize the fact that integral curves of smooth vector fields are unique to manifolds with boundary.

Lemma: *Suppose that M is a smooth manifold with nonempty boundary and that N is a smooth vector field over M. Suppose also that $\gamma : I \to M$ and $\sigma : J \to M$ are two integral curves of N, where $I, J \subseteq \mathbb{R}$ are intervals containing zero, with the same starting point: $\gamma(0) = \sigma(0)$. Then γ and σ agree on their common domain.*

Proof: This is proved exactly the same way as for manifolds without boundary. The whole idea is to prove uniqueness locally to any point in coordinates using the theory of ODEs (ordinary differential equations). Let's just prove that $\gamma(t)$ and $\sigma(t)$ agree on their common domain for $t \geq 0$ as the proof for $t \leq 0$ is similar. Let $b = \sup\{I\}$, $\beta = \sup\{J\}$, and $B = \min\{b, \beta\}$. Let $T \geq 0$ be the number:

$$T = \sup\left\{t_0 \geq 0 : t \leq B \quad \text{and} \quad \gamma|_{[0,t_0]} = \sigma|_{[0,t_0]}\right\}.$$

Intuitively speaking, if $T = B$ then γ and σ agree on their common domain for $t \geq 0$ and if $T < B$ then T represents the first time after zero when the two curves diverge. We need to prove that the first case happens. Let's prove this by contradiction: suppose $T < B$. By the continuity of both γ and σ we have that $\gamma(T) = \sigma(T)$. Let (U, φ) be a chart of M in a neighborhood of the point $\gamma(T) = \sigma(T)$ such that image(φ) is $\mathbb{R}^n$ if φ is an interior chart and $\mathbb{H}^n$ if it's a boundary chart (the last condition is merely for convenience/taste). Let $\varepsilon > 0$ be sufficiently small so that $T + \varepsilon < B$ and both γ and σ map the interval $\mathfrak{I} = (T - \varepsilon, T + \varepsilon) \cap I \cap J$ into U (the first condition isn't really necessary). Then we can consider the local coordinate representations $\hat{\gamma}, \hat{\sigma} : \mathfrak{I} \to \mathbb{R}^n$ of γ and σ given by:

$$\hat{\gamma} = \varphi \circ \gamma|_{\mathfrak{I}} \quad \text{and} \quad \hat{\sigma} = \varphi \circ \sigma|_{\mathfrak{I}}.$$

Let $\widehat{N} = \varphi_* N$ denote the vector field N pushed down to $\mathbb{R}^n$ or $\mathbb{H}^n$. If we're working in $\mathbb{H}^n$ let us also smoothly extend $\widehat{N}$ to all of $\mathbb{R}^n$. Then we have that the above local coordinate representations of γ and σ are integral curves of $\widehat{N}$. Since $\widehat{N}$ is smooth, we have by ODE theory that $\hat{\gamma}$ and $\hat{\sigma}$ must agree on their common domain. But then that means that γ and σ agree on $[0, T+\varepsilon)$, contradicting our definition of T. Hence we must indeed have that $T = B$.

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Proof of Theorem: First we will construct the function Φ in a neighborhood of any point of the form $(0,p) \in [0,\infty) \times \partial M$ using boundary charts and then use a partition of unity to construct the desired Φ. Let (U,φ) be a boundary chart of M such that range$(\varphi) = \mathbb{H}^n$ (again all of $\mathbb{H}^n$ for convenience). Consider the pushforward $\widehat{N} = \varphi_* N$ of N down to $\mathbb{H}^n$ and let $\widehat{N}_E$ denote a smooth extension of $\widehat{N}$ to all of $\mathbb{R}^n$ ("E" stands for "extension"). Let $\theta : \mathcal{D} \to \mathbb{R}^n$ denote the flow generated by $\widehat{N}_E$. Now, let's intuitively speaking "restrict" θ to $\partial\mathbb{H}^n$. Precisely, let's do this as follows: first set $\mathcal{E} \subseteq \mathcal{D}$ to be the set

$$\mathcal{E} = \{(t,x) \in \mathcal{D} : x \in \partial\mathbb{H}^n\},$$

where it's not hard to see that this is an embedded submanifold of $\mathcal{D}$. Let's observe that $\mathcal{E}$ can also be viewed as, and in fact *is*, an open submanifold of $\mathbb{R} \times \partial\mathbb{H}^n$. Finally, let $\eta : \mathcal{E} \to \mathbb{R}^n$ denote the (smooth) restriction of θ to $\mathcal{E}$.

Having η in hand, let's concentrate on its behavior near some point of the form $(0,x) \in \mathcal{E}$. Let's choose the point $(0,0)$ for simplicity. Since η is the restriction of θ to $\partial\mathbb{H}^n$, we have that

$$\text{For each } \; i \in \{1,\dots,n-1\} \qquad d\eta_{(0,0)}\left(\left.\frac{\partial}{\partial x^i}\right|_{(0,0)}\right) = \left.\frac{\partial}{\partial x^i}\right|_{(0,0)},$$

$$d\eta_{(0,0)}\left(\left.\frac{d}{dt}\right|_{(0,0)}\right) = \left.\widehat{N}_E\right|_{(0,0)}.$$

Since the last component of $\left.\widehat{N}_E\right|_{(0,0)}$ is positive (because N inward pointing on ∂M), this is a linearly independent list. So we see that $d\eta_{(0,0)}$ is invertible. Thus there exists an open neighborhood of $(0,0)$ in $\mathcal{E}$ of the form $(-\varepsilon_\varphi, \varepsilon_\varphi) \times \hat{V}_\varphi$, where $\varepsilon_\varphi > 0$ and $\hat{V}_\varphi$ is open in $\partial\mathbb{H}^n$, on which η is a local diffeomorphism (the hat $\widehat{\ }$ is placed on $\hat{V}_\varphi$ to remind us that it's sitting in Euclidean space and not on the manifold itself). By shrinking $\varepsilon_\varphi > 0$ and $\hat{V}_\varphi$ if necessary, let's also assume that $d\eta(d/dt)$ has positive last component everywhere on $(-\varepsilon_\varphi, \varepsilon_\varphi) \times \hat{V}_\varphi$. The reason for this is to ensure that η maps $(-\varepsilon_\varphi, 0) \times \hat{V}_\varphi$ into the $x^n < 0$ region and $[0, \varepsilon_\varphi) \times \hat{V}_\varphi$ into the $x^n \geq 0$ region (i.e. $\mathbb{H}^n$).

Let's consider one more restriction. Let $\tilde{\eta} : [0, \varepsilon_\varphi) \times \hat{V}_\varphi \to \mathbb{H}^n$ be the restriction of η to $[0, \varepsilon_\varphi) \times \hat{V}_\varphi$, which we think of as an open submanifold of $[0,\infty) \times \partial\mathbb{H}^n$. I claim that $\tilde{\eta}$ is a smooth embedding onto an open subset of $\mathbb{H}^n$. It's clearly a smooth immersion since it's the restriction of η and η itself is a smooth immersion (in fact a "local diffeomorphism"). Now let's show that $\tilde{\eta}$ is a topological embedding onto an open subset of $\mathbb{H}^n$. We clearly have that η is a topological embedding onto an open subset of $\mathbb{R}^n$. Since $\tilde{\eta}$ is a restriction of η, we have that it is also a topological embedding whose image is given by:

$$\eta\left[\left[0,\varepsilon_{\varphi}\right)\times\hat{V}_{\varphi}\right]=\text{range}(\eta)\cap\mathbb{H}^n,$$

where this equality follows from the last sentence of the previous paragraph. This, notice, is an open subset of $\mathbb{H}^n$ since $\text{image}(\eta)$ is open in $\mathbb{R}^n$. So we get that $\tilde{\eta}$ is indeed a smooth embedding onto an open subset of $\mathbb{H}^n$.

Great! We are now ready to construct a “local form” of Φ using our chart φ. Consider the open subset $\left[0,\varepsilon_{\varphi}\right)\times\varphi^{-1}\left[\hat{V}_{\varphi}\right]$ of $[0,\infty)\times\partial M$ and define the map $\Phi_{\varphi}:\left[0,\varepsilon_{\varphi}\right)\times\varphi^{-1}\left[\hat{V}_{\varphi}\right]\to M$ to be

$$\Phi_{\varphi}=\varphi^{-1}\circ\tilde{\eta}\circ\left(\text{id}_{[0,\varepsilon_{\varphi})}\times(\varphi|_{\partial M})\right)$$

Explicitly Φ_{φ} maps

$$\Phi_{\varphi}(t,p)=\varphi^{-1}\circ\theta\big(t,\varphi(p)\big).$$

Let’s observe a few things about Φ_{φ_k}. First of all, notice that for any $p\in\varphi^{-1}\left[\hat{V}_{\varphi}\right]$ the curve $t\mapsto\Phi_{\varphi}(t,p)$ is an integral curve of N starting at p as the following calculation shows:

$$d\Phi_{\varphi}\left(\frac{d}{dt}\right)=d\varphi^{-1}\circ d\tilde{\eta}\circ d\left(\text{id}_{[0,\varepsilon_{\varphi})}\times(\varphi|_{\partial M})\right)\left(\frac{d}{dt}\right)=d\varphi^{-1}\circ d\tilde{\eta}\left(\frac{d}{dt}\right)=d\varphi^{-1}\circ d\theta\left(\frac{d}{dt}\right)$$
$$=d\varphi^{-1}\big(\hat{N}_E\big)=N.$$

Notice also that since Φ_{φ} is a composition of maps that are smooth embeddings onto open sets, it itself is a smooth embedding onto an open set. We will use these observations below.

Now we are ready to construct our Φ out of its local forms described above by gluing them together using a partition of unity. Let $\{(U_k,\varphi_k)\}_{k=1}^{\infty}$ be a collection of boundary charts as above such that the $\varphi_k^{-1}\left[\hat{V}_{\varphi_k}\right]$ cover ∂M for $k\in\mathbb{Z}_+$. Let $\{\psi_k:\partial M\to[0,\infty)\}_{k=1}^{\infty}$ be a smooth partition of unity subordinate to this open cover of ∂M. Let $\delta:\partial M\to\mathbb{R}_+$ be the following smooth positive function:

$$\delta=\sum_{k=1}^{\infty}\varepsilon_{\varphi_k}\psi_k$$

(this is well defined of course since the ψ_k’s are locally finite). Let $\mathcal{P}_{\delta}$ be the following open subset of $[0,\infty)\times\partial M$:

$$\mathcal{P}_{\delta}=\{(t,p):p\in\partial M\quad\text{and}\quad 0\le t<\delta(p)\}$$

Finally, define $\Phi:\mathcal{P}_{\delta}\to M$ as follows. For any $(t,p)\in\mathcal{P}_{\delta}$, let $k\in\mathbb{Z}_+$ be such $p\in\varphi_k^{-1}\left[\hat{V}_{\varphi_k}\right]$ and $\delta(p)\le\varepsilon_{\varphi_k}$. Then set

$$\Phi(t,p)=\Phi_{\varphi_k}(t,p).$$

We need to show that this is well defined. Specifically, we need to show that if $k, j \in \mathbb{Z}_+$ are such that both $p \in \varphi_k^{-1}\big[\hat{V}_{\varphi_k}\big]$, $\delta(p) \leq \varepsilon_{\varphi_k}$ and $p \in \varphi_j^{-1}\left[\hat{V}_{\varphi_j}\right]$, $\delta(p) \leq \varepsilon_{\varphi_j}$, then

$$\Phi_{\varphi_k}(t, p) = \Phi_{\varphi_j}(t, p).$$

But this merely follows from the previous lemma and the fact that both $s \mapsto \Phi_{\varphi_k}(s, p)$ for $0 \leq s < \varepsilon_{\varphi_k}$ and $s \mapsto \Phi_{\varphi_j}(s, p)$ for $0 \leq s < \varepsilon_{\varphi_j}$ are integral curves of N starting at p. So Φ is indeed well defined.

To finish, we need to show that Φ satisfies all of the desired properties stated in the theorem. First let's show that $t \mapsto \Phi(t, p)$ are integral curves of N for every $p \in \partial M$. This is of course immediate from the definition of Φ. Indeed, take any $p \in \partial M$ and let $k \in \mathbb{Z}_+$ be such $p \in \varphi_k^{-1}\big[\hat{V}_{\varphi_k}\big]$ and $\delta(p) \leq \varepsilon_{\varphi_k}$. Then we have that the curve $t \mapsto \Phi(t, p) = \Phi_{\varphi_k}(t, p)$ for $0 \leq t < \delta(p) \leq \varepsilon_{\varphi_k}$ is an integral curve of N.

Now let's show that Φ is a smooth embedding onto an open neighborhood of ∂M. Let's do this in two parts: first let's show that it's injective and then that it's a smooth embedding onto an open neighborhood of ∂M. Before we show the injectivity of Φ though, let's observe one thing about Φ:

Claim: For any $p \in \partial M$, the integral curve $t \mapsto \Phi(t, p)$ for $0 \leq t < \delta(p)$ can never hit the boundary for $t > 0$.

Proof of Claim: Take any $p \in \partial M$. Let $k \in \mathbb{Z}_+$ be such $p \in \varphi_k^{-1}\big[\hat{V}_{\varphi_k}\big]$ and $\delta(p) \leq \varepsilon_{\varphi_k}$. Then we have that $\Phi(t, p) = \Phi_{\varphi_k}(t, p)$ for all $0 \leq t < \delta(p)$. But from the construction of Φ_{φ_k} above (specifically the condition $d\tilde{\eta}(d/dt)$ has positive last component everywhere) we know that the curve $t \mapsto \Phi_{\varphi_k}(t, p)$ never hits the boundary for $t > 0$. So the same holds for Φ.

End of proof of claim

Back to proving the injectivity of Φ then. Suppose that $\Phi(t_0, p_0) = \Phi(t_1, p_1)$ for some $(t_0, p_0), (t_1, p_1) \in \mathcal{P}_\delta$. We need to show that $(t_0, p_0) = (t_1, p_1)$. Let's suppose without loss of generality that $t_0 \leq t_1$. Consider the two integral curves $\gamma : [-t_0, \delta(p_0) - t_0) \to M$ and $\sigma : [-t_1, \delta(p_1) - t_1) \to M$ of N given by:

$$\gamma(s) = \Phi(s + t_0, p_0) \quad \text{and} \quad \sigma(s) = \Phi(s + t_1, p_1)$$

By assumption they have the same starting point: $\gamma(0) = \sigma(0)$. So by the previous lemma they agree on their common domain. In particular, since $t_0 \leq t_1$ we have that $\gamma(-t_0) = \sigma(-t_0)$. In other words, we have that:

$$\Phi(0, p_0) = \Phi(t_1 - t_0, p_1).$$

Since $\Phi(0, p_0) = p_0$, we can further rewrite this as:

$$p_0 = \Phi(t_1 - t_0, p_1).$$

Now, because $p_0 \in \partial M$ this equation in fact implies that $t_1 - t_0 = 0$ since if $t_1 - t_0$ were positive then we would have that the integral curve $t \mapsto \Phi(t,p)$ intersects the boundary at the positive time $t = t_1 - t_0$, contradicting our above *Claim*. So indeed $t_1 - t_0 = 0$ and hence $t_0 = t_1$. This furthermore tells us that we can rewrite the above equation as:

$$p_0 = p_1$$

since $\Phi(0,p_1) = p_1$. So we have that $(t_0,p_0) = (t_1,p_1)$ and thus Φ is indeed injective.

Finally, let's prove that Φ is a smooth embedding onto an open neighborhood of ∂M. Take any point $(t,p) \in \mathcal{P}_\delta$. As usual, let $k \in \mathbb{Z}_+$ be such $p \in \varphi_k^{-1}\left[\hat{V}_{\varphi_k}\right]$ and $\delta(p) \leq \varepsilon_{\varphi_k}$. Then we have that $\Phi = \Phi_{\varphi_k}$ on the open neighborhood $W_k = \left[0, \varepsilon_{\varphi_k}\right) \times \varphi^{-1}\left[\hat{V}_{\varphi_k}\right] \cap \mathcal{P}_\delta$ of (t,p) in $\mathcal{P}_\delta$. Now, we showed above that Φ_{φ_k} is a smooth embedding onto an open subset. And here we have that over W_k, Φ is the restriction of Φ_{φ_k} to the open set W_k. So we have that the restriction of Φ to W_k is also a smooth embedding onto an open subset. Since $(t,p) \in \mathcal{P}_\delta$ was chosen arbitrarily, with this we've shown that Φ is a smooth embedding in an open neighborhood of any point in $\mathcal{P}_\delta$ onto an open subset. This combined with the fact that Φ is injective finally gives us that Φ is indeed a smooth embedding onto an open set. Since $\Phi(0,p) = p$, the image of Φ is clearly an open neighborhood of ∂M. This proves the theorem.

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