Haim's Miscellaneous Notes

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2 Notations and Conventions

Notation 2.1: For any integer $n \geq 1$, $\mathbb{H}^n$ denotes the upper-half space of $\mathbb{R}^n$:

$$\mathbb{H}^n = \{x \in \mathbb{R}^n : x^n \geq 0\}$$

Notation 2.2: The notation $\mathbb{R}_+$ denotes the set of positive real numbers:

Convention 2.3: I use the Einstein summation convention extensively here.

Notation 2.4: Let $\Omega \subseteq \mathbb{R}^n$ be an open set. We let the following denote the following spaces of *complex-valued* functions:

1.) $C^m(\Omega)$ denotes the space of k-times continuously differentiable functions over Ω. In particular, $C^\infty(\Omega)$ denotes the space of smooth functions.

2.) $C_c^m(\Omega)$ denotes the space of k-times continuously differentiable functions over Ω *with compact support*. Sometimes $C_c^\infty(\Omega)$ is also denoted by $\mathcal{D}(\Omega)$.

3.) We let $\mathcal{S}(\mathbb{R}^n)$ denotes the space of rapidly decreasing functions:

$$\mathcal{S}(\mathbb{R}^n) = \left\{\phi \in C^\infty(\mathbb{R}^n) : \left|x^\alpha \partial^\beta \phi(x)\right| < \infty \quad \forall \alpha, \beta \in \mathcal{I}(n)\right\}.$$

This space is called the **Schwartz space**.

3 Smooth Manifolds

3.1 Immersion Pullbacks and Lie Derivatives Commute

The following is a useful result that's used in the proof of Liouville's Theorem (to be written about later).

Theorem 3.1: *Suppose that $F : M \to N$ is a smooth immersion between two smooth manifolds M and N. Suppose also that X and Y are F-related smooth vector fields over M and N respectively (i.e. $Y = dF \circ X$). Lastly, suppose that B is a smooth covariant tensor field over N of rank $k \leq dim\, M \leq dim\, N$. Then the pullback under F and Lie derivative of B commute:*

$$F^*\mathcal{L}_Y B = \mathcal{L}_X F^* B.$$

Proof: Let $m = \dim M$ and $n = \dim N$. In addition, let $A = F^*B$. Thus the equation that we want to prove is:

$$(3.2) \qquad F^*\mathcal{L}_Y B = \mathcal{L}_X A.$$

We will prove this equation in local coordinates, while choosing the most convenient ones to accomplish this task. Let (U, φ) and (V, ψ) be smooth charts of M and N respectively such that $U \subseteq F^{-1}[V]$ and such that the local coordinate representation $\hat{F} = \psi \circ F \circ \varphi^{-1}$ of F is of the form:

$$\hat{F}(x^1, \dots, x^m) = (x^1, \dots, x^m, 0, \dots, 0)$$

(such coordinates exist by the rank theorem). Let $x^1, \dots, x^m$ and $y^1, \dots, y^n$ denote the coordinate functions of φ and ψ respectively. Observe that

$$F^* dy^\mu = \begin{cases} dx^\mu & \text{if } 1 \leq \mu \leq m \\ 0 & \text{if } m+1 \leq \mu \leq n \end{cases}.$$

Observe also that at points in the image of U under F, the components of Y in the directions $\partial/\partial y^\mu$ for $\mu \in \{m+1, \dots, n\}$ are equal to zero since $Y = dF \circ X$. For this reason, the sums over μ and ν in the following calculation will only range from 1 to m. Ok, in the above chosen coordinates we have that the left-hand side of (3.2) is given by (here "ν" is sitting in the j^{th} index of A)

$$F^*\mathcal{L}_Y B = F^* \sum_{i_1,\dots,i_k=1}^{n} \left(\sum_{\mu=1}^{m} Y^\mu \frac{\partial B_{i_1,\dots i_k}}{\partial y^\mu} + \sum_{j=1}^{k} \sum_{\nu=1}^{m} \frac{\partial Y^\nu}{\partial y^{i_j}} B_{i_1,\dots,\nu,\dots,i_k} \right) dy^{i_1} \otimes \dots \otimes dy^{i_k}$$

$$= \sum_{i_1,\dots,i_k=1}^{m} \left(\sum_{\mu=1}^{m} \left[Y^\mu \frac{\partial B_{i_1,\dots i_k}}{\partial y^\mu} \right] \circ F + \sum_{j=1}^{k} \sum_{\nu=1}^{m} \left[\frac{\partial Y^\nu}{\partial y^{i_j}} B_{i_1,\dots,\nu,\dots,i_k} \right] \circ F \right) dx^{i_1} \otimes \dots \otimes dx^{i_k}$$

$$= \sum_{i_1,\dots,i_k=1}^{m} \left(\sum_{\mu=1}^{m} X^\mu \frac{\partial A_{i_1,\dots i_k}}{\partial x^\mu} + \sum_{j=1}^{k} \sum_{\nu=1}^{m} \frac{\partial X^\nu}{x^{i_j}} A_{i_1,\dots,\nu,\dots,i_k} \right) dx^{i_1} \otimes \dots \otimes dx^{i_k}.$$

As this is the right-hand side of (3.2), this proves the theorem.

3.2 b-tangent and b-cotangent bundles

In this section I put the definitions of the b-tangent and b-cotangent bundles given in the book *The Atiyah–Patodi–Singer Index Theorem*, Research Notes in Mathematics vol. 4 Section 2.2 by Richard Melrose into my own words by filling in some of the details. Please be warned that this section is meant to be terse and gives no introduction to the subject discussed as it's merely meant to formulate the definitions of b-tangent and b-cotangent bundles with all of the details in place (i.e. all of the identifications and trivializations written out explicitly). I invite the reader to first take a look at Melrose's (3 page) exposition and to consult mine while reading his account or afterwards. In this section I stray away from my own conventions and notations to match that of Melrose's.

Suppose that M is a smooth $(n+1)$-dimensional manifold with boundary. Let $\mathcal{V}_b(M)$ denote the set of all smooth vector fields over M that are tangent to the boundary ∂M at points of the boundary:

$$\mathcal{V}_b(M) = \{V : M \to TM \text{ smooth vector field over } M : V_q \in T_q(\partial M) \quad \forall q \in \partial M\}$$

(I imagine "b" stands for "boundary"). For any point $p \in M$, consider the linear subspace of finite linear combinations of such vector fields whose coefficients are smooth functions that vanish at p:

$$\mathcal{I}_p \cdot \mathcal{V}_b(M) = \left\{\sum_{i=1}^{m} f_i V_i : m \in \mathbb{Z}_+, f \in C^\infty(M) : f(p) = 0, V_i \in \mathcal{V}_b(M)\right\}$$

For any point $p \in M$, consider further the quotient vector space

$${}^bT_pM = \mathcal{V}_b(M)/\mathcal{I}_p \cdot \mathcal{V}_b(M),$$

whose elements I'll often write as $[V]_p$ which denotes the equivalence class of the vector field $V \in \mathcal{V}_b(M)$ under this quotient. The union of all such vector spaces over $p \in M$ is what we call the ***b*-tangent bundle**:

$${}^bTM = \coprod_{p \in M} {}^bT_pM$$

("$\coprod$" means "disjoint" union; you could equally well use "$\cup$" here in my opinion). We proceed to cover this with a set of smoothly compatible local trivializations to show that this is indeed a (smooth) vector bundle of rank $(n+1)$. We do this in steps:

First let's concentrate on the interior: let $(U, \varphi = (x^i))$ be local coordinates of M^{int} and let $\psi \in C^\infty(M)$ be a bump function supported in U that is identically one over an open set $\mathcal{U} \subseteq U$. For every $p \in \mathcal{U}$, consider the map $\Phi_p : {}^bT_pM \to \mathbb{R}^{n+1}$ given by

$$\Phi_p[V]_p = \left(V_p^1, \dots, V_p^{n+1}\right)$$

where V^i represent the components of V with respect to the coordinate frame $(\partial/\partial x^i)$. We leave to the reader to show that is well-defined, linear, and that it has an inverse explicitly given by

$$\Phi_p^{-1}(v^1, \dots, v^{n+1}) = \left[v^i \psi \frac{\partial}{\partial x^i}\right]_p.$$

We declare $(p, [V]_p) \in {}^bTM \longmapsto (p, \Phi_p[V]_p)$ to be a local trivialization of ${}^bTM$ over $\mathcal{U}$.

Now let's take a look at the boundary. Let $(U, \varphi = (x, y^1, \dots, y^n))$ be local coordinates of M such that $\{x = 0\} \subseteq \partial M$ and let $\psi \in C^\infty(M)$ be a bump function supported in U that is identically one over an open set $\mathcal{U} \subseteq U$. Since in these coordinates any vector field in $\mathcal{V}_b(M)$ must be of the form

$$ax \frac{\partial}{\partial x} + b^\lambda \frac{\partial}{\partial y^\lambda}$$

for some smooth functions $a, b^1, \dots, b^n$, we are guided to define the following trivializations of ${}^pTM$ over $\mathcal{U}$. For every $p \in \mathcal{U}$, consider the map $\Phi_p : {}^bT_pM \to \mathbb{R}^{n+1}$ given by

$$\Phi_p[V]_p = \left(\lim_{z \to p}\left(\frac{V^x(z)}{x(z)}\right), V_p^1, \dots, V_p^n\right)$$

where $V^x, V^1, \dots, V^n$ represent the components of V with respect to the coordinate frame $(\partial/\partial x, \partial/\partial y^1, \dots, \partial/\partial y^1)$. We leave to the reader to show that is well-defined, linear, and that it has an inverse explicitly given by

$$\Phi_p^{-1}(\alpha, b^1, \dots, b^n) = \left[\alpha \psi x \frac{\partial}{\partial x} + b^\lambda \psi \frac{\partial}{\partial y^\lambda}\right]_p.$$

As before, we declare $(p, [V]_p) \in {}^bTM \longmapsto (p, \Phi_p[V]_p)$ to be a local trivialization of ${}^bTM$ over $\mathcal{U}$.

We leave to the reader to show that all of the trivializations above are smoothly compatible. Hence indeed ${}^bTM$ is a smooth vector bundle.

We note that there is an important bundle homomorphism $F : {}^bTM \to TM$ given by the following. In any trivialization of ${}^bTM$ over the interior of M that we constructed above, F is given by

$$F\left(p, (v^1, \dots, v^{n+1})\right) = v^i \left.\frac{\partial}{\partial x^i}\right|_p.$$

In any trivialization of ${}^bTM$ near the boundary of M constructed above, F is given by

$$F\left(p, (\alpha, b^1, \dots, b^n)\right) = \alpha x(p) \left.\frac{\partial}{\partial x}\right|_p + b^i \left.\frac{\partial}{\partial y^i}\right|_p.$$

We leave it to the reader to show that this F is well-defined (i.e. the value of F doesn't depend on the trivialization we used: it follows almost by definition). The reason F is important is that, as it's not hard to check, it maps all smooth section of ${}^bTM$ bijectively onto $\mathcal{V}_b(M)$. This is the identification $\mathcal{V}_b(M) \cong C^\infty(M;\ {}^bTM)$ that Melrose is constructing in the mentioned section of his book.

Having the b-tangent bundle ${}^bTM$, we get the existence of its (smooth) dual bundle which we call the ***b*-cotangent bundle** and denote it by ${}^bT^*M$. By linear algebra we get a pullback map $F^* : T^*M \to {}^bT^*M$ mentioned in Exercise 2.6 of that section in his book given by

$$F^*(\beta)(e) = \beta\big(F(e)\big) \qquad \forall \beta \in T_p^*M \ \ \forall e \in {}^bT_pM \ \ \forall p \in M.$$

We leave to the reader to show that this is an isomorphism at any fixed $p \in M^{\text{int}}$ (hint: look at any trivialization of ${}^bTM$ over the interior of M mentioned above near p).

It's of interest to compute the inverse image under F^* of a coframe of ${}^bTM$ over the interior of M generated by trivializations of ${}^bTM$ near the boundary of M that we constructed above. Precisely, let $(U, \varphi = (x, y^1, \dots, y^n))$ be local coordinates of M such that $\{x = 0\} \subseteq \partial M$, let $\psi \in C^\infty(M)$ be a bump function supported in U that is identically one over an open set $\mathcal{U} \subseteq U$, and consider the local trivialization of ${}^bTM$ over $\mathcal{U}$ constructed from these as we did above. Let $(l, r^1, \dots, r^n)$ denote the coframe dual to the natural frame of this trivialization (i.e. the frame $p \mapsto [x\partial/\partial x]_p$ and $p \mapsto \left[\partial/\partial y^\lambda\right]_p$ over $p \in \mathcal{U}$). Then observe that for any $p \in \mathcal{U} \cap M^{\text{int}}$,

$$F^*\big(dx|_p\big)(\alpha, b^1, \dots, b^n) = dx|_p\left(\alpha x(p)\frac{\partial}{\partial x}\bigg|_p + b^i\frac{\partial}{\partial y^i}\bigg|_p\right) = \alpha x(p) = x(p)l(\alpha, b^1, \dots, b^n),$$

$$F^*\left(dy^\lambda\big|_p\right)(\alpha, b^1, \dots, b^n) = dy^\lambda\big|_p\left(\alpha x(p)\frac{\partial}{\partial x}\bigg|_p + b^i\frac{\partial}{\partial y^i}\bigg|_p\right) = b^\lambda = r^\lambda(\alpha, b^1, \dots, b^n).$$

and so

$$F^*\left(\frac{1}{x}dx\right) = l,$$

$$F^*\big(dy^\lambda\big) = r^\lambda.$$

This is what Melrose asks the reader to show in equation (2.7) in his book.

3.3 Pullback of Smooth Vector Bundles and Connections

In this note I'd like to define the pullback of smooth vector bundles and connections over them. Suppose that $F : M \to N$ is a smooth map between two smooth manifolds possibly with boundary. Suppose also that $\pi : E \to N$ is a smooth rank k vector bundle over N. We define the **pullback bundle** $\pi' : F^*E \to M$ to be the following smooth rank k vector bundle over M.

We start by letting F^*E denote the following set:

$$F^*E = \{(x, v) : x \in M, v \in \pi^{-1}[F(x)]\}.$$

In other words, we take every fiber of E (a vector space) and attach it to every point in the corresponding preimage of F. We define the projection $\pi' : F^*E \to M$ simply as $(x, v) \mapsto x$. Next let's describe the smooth local trivializations of F^*E, after which it will be clear that $\pi' : F^*E \to M$ is indeed a smooth rank k vector bundle. Take any smooth frame (b_i) for E over some open set $V \subseteq N$, and let $U = F^{-1}[V]$. We declare $(x, v) \mapsto (x, v^i)$, where v^i are the components of v with respect to (b_i), to be smooth local trivializations of F^*E over $\pi'^{-1}[U]$. We need only check that if (b_i') was another smooth frame over an open set $V' \subseteq N$ with $V \cap V' \neq \emptyset$, then the transition matrix between these two trivializations of F^*E over $\pi'^{-1}[U \cap U']$ (where $U' = F^{-1}[V']$) is smooth. But the transition matrix Φ between the frames (b_i) and (b_i') is smooth because the frames are smooth, and so the mentioned transition matrix between the two trivializations of F^*E is the smooth matrix $\Phi \circ F$. Hence indeed $\pi' : F^*E \to M$ is a smooth rank k vector bundle.

Next suppose that ∇ is a connection on E. Let TM and TN denote the tangent spaces of M and N respectively, and let $\Gamma(TM)$, $\Gamma(TN)$, $\Gamma(F^*E)$ and $\Gamma(E)$ denote the space of smooth sections of TM, TN, F^*E, and E respectively. The **pullback connection** $F^*\nabla$ on F^*E is the unique map of the form $\Gamma(TM) \times \Gamma(F^*E) \to \Gamma(F^*E)$ that satisfies

$$(3.3) \qquad (F^*\nabla)_X(F^*e) = F^*\big(\nabla_{DF(X)}e\big) \quad \forall X \in \Gamma(TM) \quad \forall e \in \Gamma(E).$$

where F^*e denotes $e \circ F$ and similarly with the right-hand side. We need to show that such a connection exists and is unique. First observe that since in a neighborhood of any point of M we can form a frame of F^*E to be of the form (F^*b_j) where (b_j) is a frame of E, such a pullback connection must be unique. Hence we only need to prove existence. We do this as follows. Take any smooth frame (b_i) for E over some open set $V \subseteq N$, let $U = F^{-1}[V]$, and consider the local trivialization of F^*E over $\pi'^{-1}[U]$ as constructed above. For any $X \in \Gamma(TM)$ and any $r \in \Gamma(F^*E)$, over U we define the pullback connection to be the operator (multilinear over $\mathbb{C}$)

$$(3.4) \qquad (F^*\nabla)_X(r) = X\big(r^i\big)F^*b_i + r^i F^*\nabla_{DF(X)}(b_i),$$

where r^i are the components of r with respect to (F^*b_i). Due to the uniqueness observation made in the previous paragraph, it will follow that this definition is well defined (i.e. independent of the (b_i) that we choose) if we show that it satisfies (3.3) above. Hence, take any $X \in \Gamma(TM)$ and take any $e \in \Gamma(E)$ which we write component wise as $e = e^i b_i$. Then by (3.4) we have that

$$(F^*\nabla)_X(F^*e) = X\big(e^i \circ F\big)F^*b_i + \big(e^i \circ F\big)F^*\nabla_{DF(X)}(b_i)$$

$$= \big[DF(X)\big(e^i\big) \circ F\big]F^*b_i + \big(e^i \circ F\big)F^*\nabla_{DF(X)}(b_i) = F^*\big[DF(X)e^i b_i + e^i\nabla_{DF(X)}(b_i)\big]$$

$$= F^*\nabla_{DF(X)}e.$$

So indeed our definition (3.4) satisfies (3.3).

Finally, we leave it to the reader to check that $F^*\nabla$ satisfies the required conditions of a connection (e.g. linearity in "e" over $\mathbb{R}$, etc.). Thus indeed $F^*\nabla$ is a connection on the pullback bundle F^*E that satisfies (3.3).

4 Real Analysis

4.1 Compactly Supported Smooth Functions are Dense in Sobolev Spaces

In Friedlander and Joshi's book *Introduction to the Theory of Distributions (2nd Ed)*, the authors prove that $C_c^\infty(\mathbb{R}^n)$ is dense in the Sobolev space $H^s(\mathbb{R}^n)$ when s is a nonnegative integer. However, they don't prove the analogous result when "s" is allowed to be any real number. A friendly postdoc from UC Santa Cruz pointed out that this general case $s \in \mathbb{R}$ follows quickly from the former as follows.

We take the perspective that the Sobolev spaces $H^s(\mathbb{R}^n)$ for $s \in \mathbb{R}$ are defined as follows:

$$H^s(\mathbb{R}^n) = \left\{u \in \mathcal{S}'(\mathbb{R}^n) : (1 + |\xi|^2)^{s/2}\hat{u}(\xi) \in L^2(\mathbb{R}^n)\right\},$$

where $\mathcal{S}'(\mathbb{R}^n)$ is the space of tempered distributions and " $\hat{}$ " denotes the Fourier transform. The norm in this space is

$$\|u\|_s = \left\|(1 + |\xi|^2)^{s/2}\hat{u}(\xi)\right\|_{L^2(\mathbb{R}^n)}.$$

Here we will take the perspective that we already know that the Schwartz functions are dense in all of the $H^s(\mathbb{R}^n)$ spaces for $s \in \mathbb{R}$ and that $C_c^\infty(\mathbb{R}^n)$ is dense in $H^k(\mathbb{R}^n)$ whenever k is a nonnegative integer because these two facts are already proven in the above-mentioned book. The proof of what we want here then becomes very short:

Theorem 4.1: *For any $s \in \mathbb{R}$, $C_c^\infty(\mathbb{R}^n)$ is dense in $H^s(\mathbb{R}^n)$.*

Proof: Pick any $s \in \mathbb{R}$. Take any $u \in H^s(\mathbb{R}^n)$. Fix any $\varepsilon > 0$. We will prove the theorem by producing a $\phi \in C_c^\infty(\mathbb{R}^n)$ such that $\|u - \phi\|_s < \varepsilon$. Since Schwartz functions are dense in $H^s(\mathbb{R}^n)$, there exists a Schwartz function $\psi \in \mathcal{S}(\mathbb{R}^n)$ such that $\|u - \psi\| < \varepsilon/2$.

Now, let k be any nonnegative integer bigger than or equal to s. Since $C_c^\infty(\mathbb{R}^n)$ is dense in $H^k(\mathbb{R}^n)$, there exists a sequence $\left\{\phi_j \in C_c^\infty(\mathbb{R}^n) : j \in \mathbb{Z}_+\right\}$ such that $\phi_j \to \psi$ in $H^k(\mathbb{R}^n)$. Since the inclusion $H^k(\mathbb{R}^n) \to H^s(\mathbb{R}^n)$ is continuous, we have that $\phi_j \to \psi$ in $H^s(\mathbb{R}^n)$ as well. Let $\phi = \phi_j$ for some $j \in \mathbb{Z}_+$ such that $\left\|\phi_j - \psi\right\|_s < \varepsilon/2$. By the triangle inequality, we then have that $\|u - \phi\|_s < \varepsilon$, which of course proves the theorem.

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4.2 Completion of Pre-Hilbert Spaces and Normed Vector Spaces

Here I want to discuss how pre-Hilbert spaces and normed vector spaces are completed with respect to their norm. Nothing here is profound, this is simply an exercise whose solution I want to be readily available to me, and so I write it up here.

Definition 4.2: A **pre-Hilbert space** is a vector space V paired with a function $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{C}$ satisfying:

1. $\langle av_1 + bv_2, w\rangle = a\langle v_1, w\rangle + b\langle v_2, w\rangle$ for any $a, b \in \mathbb{C}$ and any $v_1, v_2, w \in V$.
2. $\langle v, aw_1 + bw_2\rangle = \bar{a}\langle v, w_1\rangle + \bar{b}\langle v, w_2\rangle$ for any $a, b \in \mathbb{C}$ and any $v, w_1, w_2 \in V$.
3. $\langle v, w\rangle = \overline{\langle w, v\rangle}$ for any $v, w \in V$.

Note that (2) actually follows from (1) and (3). For any $v \in V$, we define its norm as $\|v\| = \langle v, v\rangle^{1/2}$.

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We don't require pre-Hilbert spaces to be complete. Rather we can "complete" pre-Hilbert space in the way that the following theorem states.

Theorem 4.3: *Suppose that $(V, \langle\cdot,\cdot\rangle)$ is a pre-Hilbert space. Define a class of equivalence on Cauchy sequences in V as follows: we say that two Cauchy sequences $\{u_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ in V are equivalent if $u_n - v_n \to 0$ as $n \to \infty$. For any Cauchy sequence $\{u_n\}_n$, we let $[\{u_n\}_n]$ denote its class of equivalence.*

1. *Let $\mathcal{H}$ denote the set of all classes of equivalences of Cauchy sequences in V and define addition and multiplication by scalars in $\mathcal{H}$ as*

$$[\{u_n\}_n] + [\{v_n\}_n] = [\{u_n + v_n\}_n] \qquad \forall [\{u_n\}_n], [\{v_n\}_n] \in \mathcal{H},$$
$$a[\{u_n\}_n] = [\{au_n\}_n] \qquad \forall a \in \mathbb{C} \ \ \forall [\{u_n\}_n] \in \mathcal{H}.$$

   These operations are well defined that turn $\mathcal{H}$ into a vector space.

2. *Define the function $\langle\cdot,\cdot\rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ by*

$$\langle [\{u_n\}_n], [\{v_n\}_n]\rangle = \lim_{n\to\infty} \langle u_n, v_n\rangle$$

   (we use context to differentiate $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{C}$ and $\langle\cdot,\cdot\rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$). Then this function $\langle\cdot,\cdot\rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ is well defined and $\mathcal{H}$ paired with it is a Hilbert space. Note that we of course get a norm on $\mathcal{H}$ as well.

Proof: We omit the proof of (1). First let's show that $\langle\cdot,\cdot\rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ is well-defined. Take any Cauchy sequences $\{u_n\}_n, \{v_n\}_n, \{\tilde{u}_n\}_n, \{\tilde{v}_n\}_n$ such that $[\{u_n\}] = [\{\tilde{u}_n\}_n]$ and $[\{v_n\}] = [\{\tilde{v}_n\}_n]$. We need to show that the limits $\lim_n \langle u_n, v_n\rangle$ and $\lim_n \langle \tilde{u}_n, \tilde{v}_n\rangle$ exist and are equal. Observe that

$$\left|\langle u_k, v_k\rangle - \langle u_j, v_j\rangle\right| \leq \left|\langle u_k - u_j, v_k\rangle\right| + \left|\langle u_j, v_k - v_j\rangle\right| \leq \left\|u_k - u_j\right\|\|v_k\| + \left\|u_j\right\|\left\|v_k - v_j\right\|.$$

We have that both $u_n - \tilde{u}_n \to 0$ and $v_n - \tilde{v}_n \to 0$ as $n \to \infty$. Furthermore, both $\|u_n\|$ and $\|v_n\|$ are bounded for all $n \in \mathbb{Z}$ since $\{u_n\}_n$ and $\{v_n\}_n$ are Cauchy. Hence the right-hand side of the above inequality goes to zero as $k, j \to \infty$ and so the sequence $\{\langle u_n, v_n\rangle\}_n$ converges. The same thing holds for $\{\langle \tilde{u}_n, \tilde{v}_n\rangle\}_n$. Now

$$|\langle u_n, v_n\rangle - \langle \tilde{u}_n, \tilde{v}_n\rangle| = |\langle u_n - \tilde{u}_n, v_n\rangle| + |\langle \tilde{u}_n, v_n - \tilde{v}_n\rangle| \leq \|u_n - \tilde{u}_n\|\|v_n\| + \|\tilde{u}_n\|\|v_n - \tilde{v}_n\|.$$

By reasoning as before, the right-hand side of the above inequality goes to zero as $n \to \infty$ and so indeed the limits $\lim_n\langle u_n, v_n\rangle$ and $\lim_n\langle \tilde{u}_n, \tilde{v}_n\rangle$ are equal.

Next let's prove that $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ is complete and hence a Hilbert space. Take any Cauchy sequence

$$\left\{\left[\left\{u_{k,n}\right\}_n\right]\right\}_k \tag{4.4}$$

in $\mathcal{H}$ (i.e. a Cauchy sequence of classes of equivalences of Cauchy sequences in V). We need to show that this sequence converges to something in $\mathcal{H}$. We do this by a standard diagonalization argument: for every $m \in \mathbb{Z}_+$ let n_m be such that

$$\left\|u_{m,k} - u_{m,j}\right\| < \frac{1}{m} \qquad \forall k, j \geq n_m, \tag{4.5}$$

and consider the sequence $\left\{u_{m,n_m}\right\}_m$ in V. We claim that it is Cauchy. To see why, observe that since the sequence (4.4) is Cauchy we have that for any $\varepsilon > 0$ there exists an $N_\varepsilon > 0$ such that

$$\left\|\left[\left\{u_{k,n}\right\}_n\right] - \left[\left\{u_{j,n}\right\}_n\right]\right\| < \varepsilon \qquad \forall k, j \geq N_\varepsilon$$

This is equivalent to the statement that

$$\lim_n \left\|u_{k,n} - u_{j,n}\right\| < \varepsilon \qquad \forall k, j \geq N_\varepsilon. \tag{4.6}$$

Now, take any $\varepsilon > 0$ and consider $m, \tilde{m}$ such that $1/m\,, 1/\tilde{m} < \varepsilon$ and $m, \tilde{m} \geq N_\varepsilon$. Then for any $R > 0$

$$\left\|u_{m,n_m} - u_{\tilde{m},n_{\tilde{m}}}\right\| \leq \left\|u_{m,n_m} - u_{m,R}\right\| + \left\|u_{m,R} - u_{\tilde{m},R}\right\| + \left\|u_{\tilde{m},R} - u_{\tilde{m},n_{\tilde{m}}}\right\|.$$

By (4.5) we have that for big enough R the first and third terms on the right-hand side here are less than $1/m$ and $1/\tilde{m}$ respectively. By (4.6) we have that for big enough R the second term on the right is less that ε. Hence for big enough R the right-hand side is less than 3ε. In other words,

$$\left\|u_{m,n_m} - u_{\tilde{m},n_{\tilde{m}}}\right\| < 3\varepsilon \qquad \text{if } \frac{1}{m}, \frac{1}{\tilde{m}} < \varepsilon,\ \ m, \tilde{m} \geq N_\varepsilon. \tag{4.7}$$

This proves that $\left\{u_{m,n_m}\right\}_m$ is indeed Cauchy. We finish the proof by showing that the sequence (4.4) converges to $\left[\left\{u_{m,n_m}\right\}_m\right]$ in $\mathcal{H}$. Take any $\varepsilon > 0$ and consider k such that $1/k < \varepsilon$ and $k \geq N_\varepsilon$. Then

$$\left\|\left[\left\{u_{k,m}\right\}_m\right] - \left[\left\{u_{m,n_m}\right\}_m\right]\right\| = \lim_m \left\|u_{k,m} - u_{m,n_m}\right\|. \tag{4.8}$$

Let's analyze the quantity in this limit:

$$\left\|u_{k,m} - u_{m,n_m}\right\| \leq \left\|u_{k,m} - u_{k,n_k}\right\| + \left\|u_{k,n_k} - u_{m,n_m}\right\|$$

By (4.5) the first term on the right-hand side is less than $1/k$ for large enough m. By (4.7) the second term is bounded by 3ε for large enough m. Hence the limit in (4.8) is less than 4ε. This proves that indeed the sequence (4.4) converges to $\left[\left\{u_{m,n_m}\right\}_m\right]$.

■

Theorem 4.9: *Adopt the context of the previous theorem. Then the map $i : V \to \mathcal{H}$ given by*

$$i(v) = [\{n \mapsto v\}_n]$$

(here $\{n \mapsto v\}_n$ denotes the constant sequence) is an isometric topological embedding of V in $\mathcal{H}$ as a dense subset.

Proof: We omit proving that that i is isometric and a topological embedding. To prove that $i[V]$ is dense in $\mathcal{H}$, take any point $[\{u_n\}_n] \in \mathcal{H}$ and observe that the sequence $\{i(u_k)\}_k$ converges to this point since

$$\lim_{k\to\infty} \|i(u_k) - [\{u_n\}_n]\| = \lim_{k\to\infty} \|[\{n \mapsto u_k\}_n] - [\{u_n\}_n]\| = \lim_{k\to\infty} \lim_{n\to\infty} \|u_k - u_n\| = 0.$$

■

Note 4.10: All of the above holds true if one replaces "pre-Hilbert space" with "normed vector space" and "Hilbert space" with "Banach space" and one changes all mentioned of the inner product "$\langle\cdot,\cdot\rangle$" with simply the norm "$\|\cdot\|$."

5 References

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