Haim's Notes About

Two Dimensional Compact Simple Riemannian Manifolds are Boundary Distance Rigid

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Notations and Conventions

Convention: All manifolds are without boundary by default (i.e. unless stated otherwise).

Convention: All appropriate structures/maps are assumed to be smooth (C^∞) unless stated otherwise.

Convention: In a topological space, all neighborhoods are open unless stated otherwise.

Convention: The symbol $\mathbb{R}_+$ denotes the set of positive real numbers: $\mathbb{R}_+ = \{x \in \mathbb{R} : x > 0\}$.

Definition: A **regular domain** is a smooth properly embedded submanifold with boundary of codimension zero.

Notation: If (M, g) is a Riemannian manifold possibly with boundary, then $\Omega(M)$ denotes its unit sphere bundle.

Notation: Suppose that (M, g) is a Riemannian manifold with boundary and let $\nu : \partial M \to TM$ denote the inward pointing unit normal vector field at the boundary. Then the following denote:

$$\partial_+\Omega(M) = \{(x, w) \in \Omega(M) : \langle \nu, w \rangle \geq 0\},$$

$$\partial_0\Omega(M) = \{(x, w) \in \Omega(M) : \langle \nu, w \rangle = 0\},$$

$$\partial_-\Omega(M) = \{(x, w) \in \Omega(M) : \langle \nu, w \rangle \leq 0\}.$$

The set $\partial_0\Omega(M)$ is also called the **glancing region**.

Definition: If (M, g) is a compact Riemannian manifold with boundary, then $[-\tau_-(x, v), \tau_+(x, v)]$ denotes the domain of $\gamma_{x,v}$ where $(x, v) \in SM$. We also write $\tau(x, v) = \tau_+(x, v)$, which is called the **exit time function**.

Commentary

A Boundary Value Problem on the Unit Sphere Bundle (Page 1097)

I feel the need to write down some calculations as I read this article, so here's one:

Theorem: *Suppose that (M,g) is a Riemannian manifold with boundary. Let $(S,\tilde{g})$ be a Riemannian manifold (without boundary) that contains M as a regular domain such that the inclusion $M \hookrightarrow N$ is an isometry. Let $\varphi_t : \Omega(S) \times \mathbb{R} \to \Omega(S)$ denote geodesic flow on $\Omega(S)$ and let $\mathcal{H} = (d/dt)\varphi_t|_{t=0}$ denote the geodesic vector field over $\Omega(S)$. Then for any $f \in C^\infty\big(\Omega(S)\big)$, a solution to the transport equation*

$$\mathcal{H}u = -f, \quad u|_{\partial_-\Omega(M)} = 0$$

is given by

$$u^f(x,\xi) = \int_0^{\tau(x,\xi)} f\big(\varphi_t(x,\xi)\big)dt, \quad (x,\xi) \in \Omega(M).$$

Proof: This is merely a computation. Pick any $(x,\xi) \in \Omega(M)$. Then for any appropriate $\Delta t \in \mathbb{R}$ we have that (in the second equality below I use the fundamental properties of flows)

$$u^f \circ \varphi_{\Delta t}(x,\xi) = \int_0^{\tau\left(\varphi_{\Delta t}(x,\xi)\right)} f\Big(\varphi_t\big(\varphi_{\Delta t}(x,\xi)\big)\Big)dt = \int_0^{\tau(x,\xi)-\Delta t} f\big(\varphi_{t+\Delta t}(x,\xi)\big)dt$$

$$= \int_{\Delta t}^{\tau(x,\xi)} f\big(\varphi_s(x,\xi)\big)ds.$$

Thus

$$\mathcal{H}u^f\big|_{(x,\xi)} = \frac{d}{dt}\Big(u^f \circ \varphi_t(x,\xi)\Big)\Big|_{t=0} = \lim_{\Delta t \to 0}\left(\frac{1}{\Delta t}\left[u^f \circ \varphi_{\Delta t}(x,\xi) - u^f(x,\xi)\right]\right)$$

$$= -\lim_{\Delta t\to 0}\frac{1}{\Delta t}\int_0^{\Delta t} f\big(\varphi_t(x,\xi)\big)dt = f\big(\varphi_0(x,\xi)\big) = f(x,\xi).$$

So u^f indeed satisfies the property that $\mathcal{H}u^f = f$. The boundary condition that $u^f = 0$ over $\partial_-\Omega(M)$ holds trivially since $\tau(x,\xi) = 0$ over $(x,\xi) \in \partial_-\Omega(M)$.