Haim's Notes About

X-Ray Transform and Boundary Rigidity for Asymptotically Hyperbolic Manifolds

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1 Page 2858 (PDF page 3) Metric on the b-cotangent bundle

> $\partial\overline{M}$, if $(\rho, y_1, \ldots, y_n)$ are local coordinates near $\partial\overline{M}$. The dual metric to g, viewed as a metric on ${}^bT^*\overline{M}|_M$, extends smoothly to $\overline{M}$ but degenerates

This metric on ${}^bT^*M|_M$ has the following explicit form near the boundary. Adopt the notation at the end of the section on the b-cotangent bundle in my "Miscellaneous Notes" where the boundary coordinates $(x, y^1, \ldots, y^n)$ are also chosen so that the metric g is in normal form:

$$g = \frac{dx^2 + h_{\mu\nu} dy^\mu dy^\nu}{x^2},$$

where $h_{\mu\nu}$ are smooth functions (that can depend on the x direction as well). Its associated matrix is

$$g \sim \frac{1}{x^2}\begin{bmatrix} 1 & \vec{0} \\ \vec{0} & \begin{bmatrix} h_{11} & \cdots & h_{1n} \\ \vdots & \ddots & \vdots \\ h_{n1} & \cdots & h_{nn} \end{bmatrix} \end{bmatrix}.$$

The metric on T^*M in these coordinates is then given by

$$g \sim x^2\begin{bmatrix} 1 & \vec{0} \\ \vec{0} & \begin{bmatrix} h^{11} & \cdots & h^{1n} \\ \vdots & \ddots & \vdots \\ h^{n1} & \cdots & h^{nn} \end{bmatrix} \end{bmatrix}.$$

The associated metric on ${}^bT^*\overline{M}\big|_M$ over these coordinates is explicitly given by (see my mentioned notes for $l, r^1, \dots, r^n$)

$$\begin{aligned} {}^b g(\alpha_x l + \alpha_\gamma r^\gamma, \beta_x l + \beta_\lambda r^\lambda) &= g\left(\alpha_x \frac{1}{x} dx + \alpha_\gamma dy^\gamma, \beta_x \frac{1}{x} dx + \beta_\lambda dy^\lambda\right) \\ &= \alpha_x \beta_x + x^2 h^{\gamma\lambda} \alpha_\gamma \beta_\lambda. \end{aligned}$$

The associated matrix for this is of course:

$${}^b g \sim \begin{bmatrix} 1 & \vec{0} \\ \vec{0} & x^2\begin{bmatrix} h^{11} & \cdots & h^{1n} \\ \vdots & \ddots & \vdots \\ h^{n1} & \cdots & h^{nn} \end{bmatrix} \end{bmatrix}.$$

Notice that this is indeed smooth all the way to the boundary but it degenerates (i.e. doesn't remain positive definite) at the boundary.

2 Page 2865 (PDF page 10) Inward/Outward Pointing Vectors in the *b*-Cotangent Bundle

> It is easily verified that the function $\xi \mapsto \overline{\xi_0}$ is an invariant on ${}^bT^*\overline{M}|_{\partial\overline{M}}$, i.e. it is independent of the choice of coordinates (ρ, y). In particular, the

Let's show that this map is indeed invariant of the choice of coordinates. Take any point $p_0 \in \partial M$ on the boundary. Let $(U, \varphi = (x, y^1, \dots, y^n))$ be local coordinates of M such that $\{x = 0\} \subseteq \partial M$, let $\psi \in C^\infty(M)$ be a bump function supported in U that is identically one over an open set $\mathcal{U} \subseteq U$ *containing* p_0, and consider the local trivialization of ${}^bTM$ over $\mathcal{U}$ constructed from these as I do in my "Miscellaneous Notes." Let $(l, r^1, \dots, r^n)$ denote the coframe dual to the natural frame of this trivialization (i.e. the frame $p \mapsto [x\partial/\partial x]_p$ and $p \mapsto [\partial/\partial y^\lambda]_p$ over $p \in \mathcal{U}$). Let $(\tilde{U}, \tilde{\varphi} = (\tilde{x}, \tilde{y}^1, \dots, \tilde{y}^n))$, $\tilde{\psi}$, $\tilde{\mathcal{U}}$, and $(\tilde{l}, \tilde{r}^1, \dots, \tilde{r}^n)$ be another set of such quantities that satisfy these same properties. What we want to prove will be shown if we can demonstrate that

$$\xi l_{p_0} + \eta_\lambda r^\lambda_{p_0} = \tilde{\xi}\tilde{l}_{p_0} + \tilde{\eta}_\lambda \tilde{r}^\lambda_{p_0} \qquad \Longrightarrow \qquad \xi = \tilde{\xi}.$$

To do this, let's simply derive the transformation law from $(\tilde{l}, \tilde{r}^\lambda)$ to (l, r^λ). Concentrating on the interior first and recalling the (bijective) identification $F^* : T^*\overline{M}\big|_M \to {}^bT\overline{M}\big|_M$ from my "Miscellaneous Notes," observe that for any element $\xi l + \eta_\lambda r^\lambda = \tilde{\xi}\tilde{l} + \tilde{\eta}_\lambda r^\lambda$ of ${}^bT^*M$ at any point of $\mathcal{U} \cap \tilde{\mathcal{U}} \cap M$,

$$\xi l + \eta_\lambda r^\lambda = F^*\left(\xi \frac{1}{x} dx + \eta_\lambda dy^\lambda\right) = F^*\left(\left(\xi \frac{1}{x}\frac{\partial x}{\partial \tilde{x}} + \eta_\lambda \frac{\partial y^\lambda}{\partial \tilde{x}}\right) d\tilde{x} + \left(\xi \frac{1}{x}\frac{\partial x}{\partial \tilde{y}^\mu} + \eta_\lambda \frac{\partial y^\lambda}{\partial \tilde{y}^\mu}\right) d\tilde{y}^\mu\right)$$

$$= \left(\xi \frac{\tilde{x}}{x}\frac{\partial x}{\partial \tilde{x}} + \eta_\lambda \tilde{x} \frac{\partial y^\lambda}{\partial \tilde{x}}\right)\tilde{l} + \left(\xi \frac{1}{x}\frac{\partial x}{\partial \tilde{y}^\mu} + \eta_\lambda \frac{\partial y^\lambda}{\partial \tilde{y}^\mu}\right)\tilde{r}^\mu = \tilde{\xi}\tilde{l} + \tilde{\eta}_\mu \tilde{r}^\mu.$$

Hence the transformation law over $\mathcal{U} \cap \tilde{\mathcal{U}} \cap M$ is

$$\tilde{\xi} = \left(\xi \frac{\tilde{x}}{x}\frac{\partial x}{\partial \tilde{x}} + \eta_\lambda \tilde{x} \frac{\partial y^\lambda}{\partial \tilde{x}}\right),$$

$$\tilde{\eta} = \xi \frac{1}{x}\frac{\partial x}{\partial \tilde{y}^\mu} + \eta_\lambda \frac{\partial y^\lambda}{\partial \tilde{y}^\mu}.$$

These actually hold at the boundary ∂M as well because it's not hard to see that all of the quantities in the above two equations extend continuously to ∂M. In fact, we have that at any boundary point, such as $p_0 \in \partial M$, the above first equation reduces to $\tilde{\xi} = \xi$. This is what we wanted to show.

3 Page 2866 (PDF page 11) The ${}^{\mathcal{L}}T\overline{M}$ smooth vector bundle

$${}^{\mathcal{L}}\mathcal{V} = \{V \in C^\infty(\overline{M}; T\overline{M}) : V|_{\partial \overline{M}} = 0 \text{ and } (\rho^{-1}V)(x) \in \mathcal{L}_x,\ x \in \partial\overline{M}\}.$$

In the usual way, ${}^{\mathcal{L}}\mathcal{V}$ can be regarded as the space of smooth sections of a smooth vector bundle ${}^{\mathcal{L}}T\overline{M}$ on $\overline{M}$. If $(\rho, y = y^1, \dots, y^n)$ are any local

Here I just want to give a quick note about what the smooth vector bundle ${}^{\mathcal{L}}T\overline{M}$ mentioned here is. It can in fact be constructed, as we will shortly, similarly to the way that Melrose constructs the b-tangent bundle in his book *The Atiyah–Patodi–Singer index theorem*, Research Notes in Mathematics vol. 4 Section 2.2. Here I will follow my exposition of Melrose's construction that I wrote up in my "Miscellaneous Notes." In fact, the following text is mostly copied and pasted from that note and tweaked so that it suits our purposes.

We start by defining the following weird notation for every $p \in M$:

$$\mathcal{I}_p \cdot {}^{\mathcal{L}}\mathcal{V} = \left\{ \sum_{i=1}^{m} f_i V_i : m \in \mathbb{Z}_+, f \in C^\infty(\bar{M}) : f(p) = 0, V_i \in {}^{\mathcal{L}}\mathcal{V} \right\}.$$

For every $p \in \bar{M}$, consider the following quotient vector space:

$${}^{\mathcal{L}}T_p\bar{M} = {}^{\mathcal{L}}\mathcal{V}/\left(\mathcal{I}_p \cdot {}^{\mathcal{L}}\mathcal{V}\right),$$

whose elements I'll often write as "$[V]_p$," which denotes the equivalence class of a vector field $V \in {}^{\mathcal{L}}\mathcal{V}$ under this quotient. Now, let ${}^{\mathcal{L}}T\bar{M}$ be the disjoint union

$${}^{\mathcal{L}}T\bar{M} = \coprod_{p\in M} {}^{\mathcal{L}}T_p\bar{M}.$$

We proceed to cover this with a set of smoothly compatible local trivializations to show that this is indeed a (smooth) vector bundle of rank $(n+1) = \dim \bar{M}$. We do this in steps:

First let's concentrate on the interior: let $(U, \varphi = (x^i))$ be local coordinates of (the interior) M and let $\psi \in C^\infty(M)$ be a bump function supported in U that is identically one over an open set $\mathcal{U} \subseteq U$. For every $p \in \mathcal{U}$, consider the map $\Phi_p : {}^{\mathcal{L}}T_p\bar{M} \to \mathbb{R}^{n+1}$ given by

$$\Phi_p[V]_p = \left(V_p^1, \dots, V_p^{n+1}\right)$$

where V^i represent the components of V with respect to the coordinate frame $(\partial/\partial x^i)$. We leave to the reader to show that is well-defined, linear, and that it has an inverse explicitly given by

$$\Phi_p^{-1}(v^1, \dots, v^{n+1}) = \left[v^i \psi \frac{\partial}{\partial x^i}\right]_p.$$

We declare $(p, [V]_p) \in {}^{\mathcal{L}}T\bar{M} \longmapsto (p, \Phi_p[V]_p)$ to be a local trivialization of ${}^{\mathcal{L}}T\bar{M}$ over $\mathcal{U}$.

Now let's take a look at the boundary. Let $(U, \varphi = (x, y^1, \dots, y^n))$ be local coordinates of $\bar{M}$ such that $\{x = 0\} \subseteq \partial M$ and $\partial/\partial x$ is orthogonal to $\partial \bar{M}$ with respect to $\rho^2 g$ for some (and hence any) boundary defining function $\rho : \bar{M} \to \mathbb{R}$ (in orange for emphasis: see below why). Let $\psi \in C^\infty(M)$ be a bump function supported in U that is identically one over an open set $\mathcal{U} \subseteq U$. It's not hard to see that in these coordinates, any vector field in ${}^{\mathcal{L}}\mathcal{V}$ must be of the form

$$ax\frac{\partial}{\partial x} + b^\lambda x^2 \frac{\partial}{\partial y^\lambda}$$

for some smooth functions $a, b^1, \dots, b^n$. Hence we are guided to define the following trivializations of ${}^{\mathcal{L}}T\bar{M}$ over $\mathcal{U}$. For every $p \in \mathcal{U}$, consider the map $\Phi_p : {}^{\mathcal{L}}T_p\bar{M} \to \mathbb{R}^{n+1}$ given by (read this carefully)

$$\Phi_p[V]_p = \left(\lim_{z\to p}\left(\frac{V^x(z)}{x(z)}\right), \lim_{z\to p}\left(\frac{V^1(z)}{x^2(z)}\right), \dots, \lim_{z\to p}\left(\frac{V^n(z)}{x^2(z)}\right)\right) \tag{3.1}$$

where $V^x, V^1, \dots, V^n$ represent the components of V with respect to the coordinate frame $(\partial/\partial x, \partial/\partial y^1, \dots, \partial/\partial y^1)$. We leave to the reader to show that is well-defined, linear, and that it has an inverse explicitly given by

$$\Phi_p^{-1}(\alpha, b^1, \dots, b^n) = \left[\alpha\psi x \frac{\partial}{\partial x} + b^\lambda \psi x^2 \frac{\partial}{\partial y^\lambda}\right]_p.$$

As before, we declare $(p, [V]_p) \in {}^{\mathcal{L}}T\overline{M} \longmapsto (p, \Phi_p[V]_p)$ to be a local trivialization of ${}^{\mathcal{L}}T\overline{M}$ over $\mathcal{U}$.

We leave to the reader to show that all of the trivializations above are smoothly compatible (the requirement $\partial/\partial x \perp \partial\overline{M}$ that we imposed in the boundary trivializations above will be needed to check the smooth compatibility of two such boundary trivializations). Hence indeed ${}^{\mathcal{L}}T\overline{M}$ is a smooth vector bundle of rank $(n+1)$.

We note that there is an important bundle homomorphism $F : {}^{\mathcal{L}}T\overline{M} \to T\overline{M}$ given by the following. In any trivialization of ${}^{\mathcal{L}}T\overline{M}$ over the interior of $\overline{M}$ that we constructed as above, F is given by

$$F\big(p, (v^1, \dots, v^{n+1})\big) = v^i \frac{\partial}{\partial x^i}\bigg|_p.$$

In any trivialization of ${}^{\mathcal{L}}T\overline{M}$ near the boundary of $\overline{M}$ as constructed above, F is given by

$$F\big(p, (\alpha, b^1, \dots, b^n)\big) = \alpha x(p) \frac{\partial}{\partial x}\bigg|_p + b^i x^2(p) \frac{\partial}{\partial y^i}\bigg|_p.$$

We leave it to the reader to show that this F is well-defined (i.e. the value of F doesn't depend on the trivialization we used). The reason F is important is that, as it's not hard to check, it maps all smooth section of ${}^{\mathcal{L}}T\overline{M}$ bijectively onto ${}^{\mathcal{L}}\mathcal{V}$. This bijection/identification is what the authors are referring to when they say "In the usual way, ${}^{\mathcal{L}}\mathcal{V}$ can be regarded as the space of smooth sections of a smooth vector bundle ${}^{\mathcal{L}}T\overline{M}$ on $\overline{M}$."

4 Page 2866 (PDF Page 11) Metric on ${}^{\mathcal{L}}T\overline{M}$

In this note I give an explicit expression for the metric on ${}^{\mathcal{L}}T\overline{M}$ induced by g near the boundary $\partial\overline{M}$.

Consider a local trivialization of ${}^{\mathcal{L}}T\overline{M}$ near the boundary $\partial\overline{M}$ of the form that we constructed in Section 3 (c.f equation (3.1) above) and adopt the notation that we were using to construct that trivialization. Suppose furthermore that when constructing that trivialization, the x in the boundary coordinates (x, y^i) used there is a geodesic boundary defining function for some representative of the conformal infinity of (M, g) (c.f. first two pages of the paper). Specifically this implies that in the coordinates (x, y^i) the metric g takes the form

$$g = \frac{dx^2 + h_{\mu\nu} dy^\mu dy^\nu}{x^2},$$

where $h_{\mu\nu}$ are smooth functions (that can depend on the x direction as well). Its associated matrix is of course

$$g \sim \frac{1}{x^2}\begin{bmatrix} 1 & \vec{0} \\ \vec{0} & \begin{bmatrix} h_{11} & \cdots & h_{1n} \\ \vdots & \ddots & \vdots \\ h_{n1} & \cdots & h_{nn} \end{bmatrix} \end{bmatrix}.$$

We should mention that we can assume that x is a geodesic boundary defining function because it satisfies the requirement we had that $\partial/\partial x$ is perpendicular to the boundary $\partial\overline{M}$ when we were constructing such a trivialization of ${}^{\mathcal{L}}T\overline{M}$.

Now, let $v, w_1, \ldots, w_n$ denote the coordinate frame in our local boundary trivialization of ${}^{\mathcal{L}}T\overline{M}$. Then, an expression for the induced metric on ${}^{\mathcal{L}}T\overline{M}$ is given by

$$\begin{aligned} {}^{\mathcal{L}}g\left(av + a^i w_i, bv + b^j w_j\right) &= g\left(ax\frac{\partial}{\partial x} + a^i x^2 \frac{\partial}{\partial y^i}, bx\frac{\partial}{\partial x} + b^j x^2 \frac{\partial}{\partial y^j}\right) \\ &= ab + a^i b^j x^2 h_{ij}. \end{aligned}$$

In matrix form,

$${}^{\mathcal{L}}g \sim \begin{bmatrix} 1 & \vec{0} \\ \vec{0} & x^2\begin{bmatrix} h_{11} & \cdots & h_{1n} \\ \vdots & \ddots & \vdots \\ h_{n1} & \cdots & h_{nn} \end{bmatrix} \end{bmatrix}.$$

Obviously this metric extends smoothly to the boundary, but it degenerates at the boundary itself.

5 Page 2866 (PDF page 11) 0-Cotangent Bundle

> are parametrized by the fibers $\partial_{\mp} S^*_x M$. As a comparison, recall that the 0-cotangent bundle is the smooth bundle ${}^0T^*\overline{M}$ over $\overline{M}$ whose fibers near the boundary have basis $\{\frac{d\rho}{\rho}, \frac{dy^i}{\rho}\}$. The 0-unit cotangent bundle is ${}^0S^*\overline{M} :=$

Here I'd like to give a brief description of what the 0-cotangent bundle is. *As I understand*, the 0-tangent bundle ${}^0T\overline{M}$ is a bundle over $\overline{M}$ all of whose smooth sections are canonically identified with the set of all smooth vector fields over $\overline{M}$ that vanish at the boundary $\partial\overline{M}$. The 0-cotangent bundle ${}^0T^*\overline{M}$ is then of course the dual bundle to ${}^0T\overline{M}$.

I would highly recommend first reading my notes on the b-tangent/cotangent bundles in my "Miscellaneous Notes" before reading this section. Referring to these said notes we construct ${}^0T\overline{M}$ exactly the same way as we did ${}^bT\overline{M}$ there (we wrote M there rather than $\overline{M}$) except with a few changes, the first of which is that we instead consider the fibers

$$ {}^0T_p\overline{M} = \mathcal{V}_0(\overline{M})/\mathcal{I}_p \cdot \mathcal{V}_0(\overline{M}), $$

where $\mathcal{V}_0(\overline{M})$ denotes the space of all smooth vector fields on $\overline{M}$ that vanish at the boundary and

$$ \mathcal{I}_p \cdot \mathcal{V}_0(\overline{M}) = \left\{ \sum_{i=1}^{m} f_i V_i : m \in \mathbb{Z}_+, f \in C^\infty(M) : f(p) = 0, V_i \in \mathcal{V}_0(\overline{M}) \right\} $$

The next change is that the equation for the *boundary* trivializations needs to be changed to

$$ \Phi_p[V]_p = \left(\lim_{z\to p} \left(\frac{V^x(z)}{x(z)} \right), \lim_{z\to p} \left(\frac{V^1(z)}{x(z)} \right), \dots, \lim_{z\to p} \left(\frac{V^n(z)}{x(z)} \right) \right) $$

and its inverse to

$$ \Phi_p^{-1}(\alpha, b^1, \dots, b^n) = \left[\alpha\psi x \frac{\partial}{\partial x} + b^\lambda \psi x \frac{\partial}{\partial y^\lambda} \right]_p . $$

We also have an analogous bundle homomorphism of the form $F : {}^0T\overline{M} \to T\overline{M}$, where the only difference is that in the boundary trivializations the equation for F is instead given by

$$ F\big(p, (\alpha, b^1, \dots, b^n)\big) = \alpha x(p) \frac{\partial}{\partial x}\bigg|_p + b^i x(p) \frac{\partial}{\partial y^i}\bigg|_p . $$

It's not hard to see that this F maps smooth sections of ${}^0T\overline{M}$ bijectively onto $\mathcal{V}_0(\overline{M})$, which is the canonical identification that we wanted. Now, ${}^0T^*\overline{M}$ is the dual bundle to ${}^0T\overline{M}$, and as in the mentioned b-tangent/cotangent bundles notes we have an analogous map $F^* : T^*\overline{M} \to {}^0T^*\overline{M}$. The only difference from there is that in the boundary trivializations the equations for F^* need to be changed to

$$ F^*\left(\frac{1}{x} dx \right) = l, $$

$$ F^*\left(\frac{1}{x} dy^\lambda \right) = r^\lambda. $$

6 Page 2868 (PDF page 13) Canonical Symplectic form on $\partial_\mp S^*M$

as elements of $\partial_\mp S^*M$. So the identification (2.2) is determined up to the map $(y,\eta) \mapsto (y, \eta \mp \mathrm{d}u(y))$ of $T^*\partial\overline{M}$. This is a symplectomorphism of $T^*\partial\overline{M}$ for each u, so it follows that each of $\partial_\mp S^*M$ has a canonical structure as a symplectic manifold, with symplectic form $\sum_i \mathrm{d}\eta_i \wedge \mathrm{d}y^i$.

I want to explain here in a bit more detail what these symplectic forms on $\partial_{\mp}S^*M$ are. First we have to understand how the maps described in (2.2) in the paper depend on the representative h of the conformal infinity. Suppose that you take another representative $\hat{h}$, whose relation to h we can always write as $\hat{h} = e^{2u}h$ for some $u \in C^\infty(\partial\overline{M}, \mathbb{R})$. Let ρ and $\hat{\rho}$ denote the geodesic boundary defining functions of h and $\hat{h}$ respectively (c.f. first two pages of the paper). The authors claim that if you choose coordinates (y^i) and $(\hat{y}^i)$ of $\partial\overline{M}$ and form the boundary coordinates (ρ, y^i) and $(\hat{\rho}, \hat{y}^i)$ of $\partial\overline{M}$, then we will have the following relation holds

$$\hat{\rho} = e^u\rho + O(\rho^2),$$

$$\hat{y}^i = y^i + O(\rho).$$

We actually just need the first equation, which let's write out a bit more explicitly in the coordinates (ρ, y^i) as (here $y = (y^1, \dots, y^n)$)

$$\hat{\rho}(\rho, y) = e^{u(\rho,y)}\rho + f(\rho, y),$$

where $u(\rho, y)$ is a smooth extension of u into the interior M and f is such that f/ρ^2 extends smoothly to the boundary. I don't know how this follows from anything described previously in the paper: I imagine you'd have to take a look at the reference "[16]" mentioned in the paper to see how. Furthermore, I don't know if the authors meant that the u and f above indeed also depend on ρ as well, but I threw that in just in case because it doesn't affect our discussion here.

Consider boundary local trivializations of ${}^bT\overline{M}$ constructed from (ρ, y^i) and $(\hat{\rho}, \hat{y}^i)$ as we did in the section on the b-tangent bundle in my "Miscellaneous notes" and let $(l, r^1, \dots, r^n)$ and $(\hat{l}, \hat{r}^1, \dots, \hat{r}^n)$ denote the coframe dual to the natural frames of these trivializations respectively (for clarification, see end of that mentioned section). Let $G : T^*\overline{M} \to {}^bT^*M$ denote the map that provides the canonical isomorphism of $T_x^*\overline{M}$ and ${}^bT_x^*\overline{M}$ whenever $x \in M$ is in the interior (c.f. end of that mentioned section). We have that the identification (2.2) mentioned in the paper is given in these trivializations and coordinates as

$$\pm l + \eta_i r^i \longmapsto \eta_i dy^i,$$

$$\pm\hat{l} + \hat{\eta}_i\hat{r}^i \longmapsto \hat{\eta}_i d\hat{y}^i.$$

Call these two maps $F_\pm : \partial_\pm S^*M \to T^*\partial\overline{M}$ and $\hat{F}_\pm : \partial_\pm S^*M \to T^*\partial\overline{M}$ respectively. We want to see how these two maps are related. To do this, let's track where $F_\pm$ maps $\pm\hat{l} + \hat{\eta}_i\hat{r}^i$. We observe that away from the boundary $\partial\overline{M}$,

$$G\left(\pm\hat{l} + \hat{\eta}_i\hat{r}^i\right) = \pm\hat{\rho}^{-1}d\hat{\rho} + \hat{\eta}_i d\hat{y}^i = \left(\pm\hat{\rho}^{-1}\frac{\partial\hat{\rho}}{\partial\rho} + \hat{\eta}_i\frac{\partial\hat{y}^i}{\partial\rho}\right)d\rho + \left(\pm\hat{\rho}^{-1}\frac{\partial\hat{\rho}}{\partial y^j} + \hat{\eta}_i\frac{\partial\hat{y}^i}{\partial y^j}\right)dy^j.$$

It's not hard to see from this equation that $F_\pm$ takes $\pm\hat{l} + \hat{\eta}_i\hat{r}^i$ to

$$\left[\lim_{\rho\to 0^+}\left(\pm\hat{\rho}^{-1}\frac{\partial\hat{\rho}}{\partial y^j} + \hat{\eta}_i\frac{\partial\hat{y}^i}{\partial y^j}\right)\right]dy^i = \left[\lim_{\rho\to 0^+}\left(\frac{\pm 1}{e^u\rho + f}\left[\frac{\partial u}{\partial y^j}e^u\rho + \frac{\partial f}{\partial y^j}\right] + \hat{\eta}_i\frac{\partial\hat{y}^i}{\partial y^j}\right)\right]dy^i$$

$$= \left(\pm\frac{\partial u}{\partial y^j} + \hat{\eta}_i \frac{\partial \hat{y}^i}{\partial y^j}\right) dy^j = \pm du + \hat{\eta}_i \partial \hat{y}^i = \pm du + \hat{F}_\pm\left(\pm \hat{l} + \hat{\eta}_i \hat{r}^i\right).$$

Hence we've arrived at that

$$F_\pm(\zeta) = \hat{F}_\pm(\zeta) \pm du, \qquad \forall \zeta \in \partial_\pm S^* M.$$

Good. So, we understand how the identification (2.2) mentioned in the paper depends on the representative of the conformal infinity. Continuing, let's see how these two maps $F_\pm$ and $\hat{F}_\pm$ pullback the canonical symplectic form on $T^*\partial\bar{M}$:

$$\omega = \sum_{i=1}^{n} d\eta_i \wedge dy^i.$$

In fact, as we're about to show, these two mentioned pullbacks are the same: $F_\pm^*\omega = \hat{F}_\pm^*\omega$, and we will demonstrate this by proving that the map $P : T^*\partial\bar{M} \to T^*\partial\bar{M}$ given by $\vartheta \mapsto \vartheta + du$ is a symplectomorphism. This is simply seen by computing $P^*\omega$ in coordinates:

$$P^*\omega = \sum_{i=1}^{n} d(\eta_i \circ P) \wedge d\left(y^i \circ P\right) = \sum_{i=1}^{n}\left(d\eta_i + \frac{\partial}{\partial y^j}\sum_{j=1}^{n}\left[\frac{\partial u}{\partial y^i}\right] dy^j\right) \wedge dy^i$$

$$= \omega + \sum_{ij=1}^{n} \frac{\partial^2 u}{\partial y^j \partial y^i} dy^j \wedge dy^i.$$

Since the second term in the last expression is zero, we indeed get that P is a symplectomorphism: $P^*\omega = \omega$. As mentioned above, this implies that $F_\pm^*\omega = \hat{F}_\pm^*\omega$.

From here we see that the pullback of the canonical symplectic form on $T^*\partial\bar{M}$ by the identification (2.2) in the paper is the same regardless of the representative of the conformal infinity that we choose. We call that pulled back symplectic form the "canonical symplectic form on $\partial_\pm S^* M$."

7 Page 2877 (PDF page 22) Equivalent Condition for Points to be Conjugate

$\left(Y(0), D_t Y(0)\right) = \mathcal{L}(\zeta) = \left(\mathrm{d}\pi(h), \mathcal{K}(v)\right)$. Two points $p, q \in M$ are said to be *conjugate points* if there exist $z \in S_p^* M$ and $T > 0$ so that $\varphi_T(z) \in S_q^* M$ and

$$\mathrm{d}\varphi_T(z).\mathcal{V}(z) \cap \mathcal{V}(\varphi_T(z)) \neq \{0\}. \tag{2.28}$$

This is equivalent to the statement that there is a normal Jacobi field along γ which vanishes at both 0 and T.

In this section I'd like to discuss why (2.28) in the paper is equivalent to p and q being conjugate. I will do things in the tangent bundle here instead, and then discuss why things transfer over to the cotangent bundle in the end.

Let $z=(x,v)$, where $v\in SM$, be the explicit expression for any point $z\in SM$. Let $\tilde{X}$ (note that it's a tilde and not a hat) denote the geodesic vector field along SM and $\tilde{\varphi}$ its geodesic flow. It's well known (and the reader can probably guess how) that in the tangent bundle we can define analogous projection and connection maps $\widetilde{d\pi}:TTM\to TM$ and $\widetilde{\mathcal{K}}:TTM\to TM$, define an analogous splitting $TSM=\mathbb{R}\tilde{X}\oplus\widetilde{\mathcal{H}}\oplus\tilde{\mathcal{V}}$ where $\mathbb{R}\tilde{X}\perp\left(\widetilde{\mathcal{H}}\oplus\tilde{\mathcal{V}}\right)$ with respect to the Sasaki metric $\tilde{G}$, and an analogous identification $\tilde{\mathcal{L}}:\widetilde{\mathcal{H}}\oplus\tilde{\mathcal{V}}\to\mathcal{Z}\oplus\mathcal{Z}$ where $\mathcal{Z}_{(x,v)}=\{w\in T_xM : w\perp v\}$. Then for any $(T,z)\in\mathbb{R}\times SM$ in the domain of $\tilde{\varphi}$, the following equation holds

$$(7.1)\qquad d\tilde{\varphi}_T(z).w=\widetilde{d\pi}\big|^{-1}_{\mathbb{R}\tilde{X}\oplus\widetilde{\mathcal{H}}(\tilde{\varphi}_T(z))}\left(J_{z,w}(T)\right)+\widetilde{\mathcal{K}}\big|^{-1}_{\tilde{\mathcal{V}}(\tilde{\varphi}_T(z))}\left(D_tJ_{z,w}(T)\right)\quad\forall w\in T_zSM$$

where $J_{z,w}$ is the Jacobi field along the geodesic $t\mapsto\tilde{\pi}\circ\tilde{\varphi}_t(z)$ satisfying $J_{z,w}(0)=\widetilde{d\pi}(w)$ and $D_tJ_{z,w}(0)=\widetilde{\mathcal{K}}(w)$. This equation is an exercise in (Paternain, Salo, & Uhlmann, 2022), the proof of which we work out here. Take any $w\in T_zSM$ and let $\alpha:(a,b)\to M$ be a smooth curve on M and $V:(a,b)\to TM$ be a smooth vector field along α such that the velocity of the curve $(\alpha,V):(a,b)\to TM$ at $s=0$ is w (technically V and (α,V) are the same thing). Then we have that

$$d\tilde{\varphi}_T(z).w=\frac{d}{ds}\bigg|_{s=0}\left(\exp_{\alpha(s)}\big(T\cdot W(s)\big),\frac{d}{dt}\bigg|_{t=T}\exp_{\alpha(s)}\big(t\cdot W(s)\big)\right).$$

Now, it shouldn't be hard to see that applying the $d/ds|_{s=0}$ derivative to the both the position and velocity component on the right-hand side gives us that

$$d\tilde{\varphi}_T(z).w=\widetilde{d\pi}\big|^{-1}_{\dots}\left(J_{z,v}(T)\right)+\widetilde{\mathcal{K}}\big|^{-1}_{\dots}\left(\frac{d}{ds}\bigg|_{s=0}\frac{d}{dt}\bigg|_{t=T}\exp_x\big(t\cdot W(s)\big)\right)$$

where "…" are written out two equations back. To see what the argument of $\widetilde{\mathcal{K}}$ here is equal to, consider coordinates of TM that are generated by normal coordinates centered at $\tilde{\pi}(\tilde{\varphi}_T(z))$, interchange the d/ds and d/dt derivatives, from which you should get that the it is $D_tJ_{z,w}(T)$. Hence follows the equation I claimed above.

It's not hard to see that this implies that the points p and q are conjugate if and only if there exists a $z\in SM$ and $T>0$ such that

$$d\tilde{\varphi}_T(z).\tilde{\mathcal{V}}(z)\cap\tilde{\mathcal{V}}\big(\tilde{\varphi}_T(z)\big)\neq\{0\}.$$

The reason the analogous statement holds on the cosphere S^*M is that the differential of the musical isomorphism "♭" bijectively maps $\tilde{\mathcal{V}}(z)$ onto $\mathcal{V}\big(z^\flat\big)$.

8 Page 2882 (PDF page 27) Details on (3.11): Adjoints of Resolvents at Zero

Proof. — First by (3.2), we have for each $f, f' \in C_c^\infty(S^*M)$ real valued,

$$\langle R_+(0)f, f'\rangle = -\langle f, R_-(0)f'\rangle, \tag{3.11}$$

In this note I'd like to fill in the details on how (3.11) in the paper follows from (3.2). We have by (3.2) that for any real valued $f, f' \in C_c^\infty(S^*M)$,

$$\langle R_+(0)f, f'\rangle = \int_{S^*M} f'(z) \int_0^\infty f \circ \varphi_t(z)dt\, dz \tag{8.1}$$

$$= \int_{\partial\_S^*M} I\left(z \mapsto f'(z) \int_0^\infty f \circ \varphi_t(z)dt\right)(w)|\mu_\partial(w)|$$

where we applied Santaló's formula (Lemma 3.6 in the paper) in the last equality here. Observe that since the argument "$z \mapsto \cdots$" in last quantity is compactly supported, it's now hard to see by definition of "I" in (3.8) that for every fixed $w \in \partial\_S^*M$ the last integrand

$$I\left(z \mapsto f'(z) \int_0^\infty f \circ \varphi_t(z)dt\right)(w) = \int_{-\infty}^\infty f' \circ \varphi_s(z_w) \int_0^\infty f \circ \varphi_t\big(\varphi_s(z_w)\big)dt\, ds \tag{8.2}$$

for any $z_w \in S^*M$ on the integral curve of $\overline{X}$ starting at w (identifying $S^*M \cong \overline{S^*M} \setminus \partial S^*M$). This last quantity is equal to

$\int_{-\infty}^\infty \int_0^\infty f' \circ \varphi_s(z_w)\, f \circ \varphi_{t+s}(z_w)dt\, ds$	Just rewrote the quantity
$\int_{-\infty}^\infty \int_{-\infty}^r f' \circ \varphi_u(z_w)\, f \circ \varphi_r(z_w)du\, dr$	Thought of the previous integral as the result of the change of variables $(u, r) = (s, t + s)$.
$\int_{-\infty}^\infty f \circ \varphi_r(z_w) \int_{-\infty}^r f' \circ \varphi_u(z_w)du\, dr$	Pulled out $f \circ \varphi_r(z_w)$ from the inside integral.
$\int_{-\infty}^\infty f \circ \varphi_r(z_w) \int_{-\infty}^0 f' \circ \varphi_{\tilde{u}+r}(z_w)d\tilde{u}\, dr$	Made the change of variables $\tilde{u} = u - r$.

$$\int_{-\infty}^{\infty} f \circ \varphi_r(z_w) \int_0^{\infty} f' \circ \varphi_{-u'}\big(\varphi_r(z_w)\big) du'\, dr$$

Made the further change of variables $u' = -\tilde{u}$.

From the resemblance between the last quantity and the right-hand side of (8.2) above and considering (8.1) we arrive at that (I changed "u'" here back to "t")

$$(8.3) \qquad \langle R_+(0) f, f' \rangle = \int_{S^*M} f(z) \int_0^{\infty} f' \circ \varphi_{-t}(z) dt\, dz.$$

By (3.2) in the paper we have that this last quantity is equal to $-\langle f, R_-(0) f' \rangle$, which proves what we wanted.

9 Page 2882 (PDF page 27) Showing that $\Pi = I^* I$

$$\langle R_+(0) f, f \rangle = -\int_{S^*M} Xu.u\mu = -\frac{1}{2} \int_{S^*M} X(u^2)\mu = \frac{1}{2} \int_{\partial_- S^*M} (If)^2 |\mu_\partial|$$

which shows $\Pi f = I^* If$ using (3.11). By Lemma 3.5 and since $If \in$

I just want to give a quick note on how $\Pi = I^* I$ over $C_c^\infty(S^*M)$ follows. From (8.1) and (8.2) above, it's easy to see that for any $f \in C_c^\infty(S^*M)$

$$\langle R_+(0) f, f \rangle = \int_{\partial_- S^*M} \int_{-\infty}^{\infty} f' \circ \varphi_s(z_w) \int_0^{\infty} f \circ \varphi_t\big(\varphi_s(z_w)\big) dt\, ds\, |\mu_\partial(w)|$$

where for each $w \in \partial_- S^*M$, $z_w \in S^*M$ denotes any point on the integral curve of $\overline{X}$ starting at w. By (8.3) above, it's not hard to see that the same formula holds if we replace "t" by "$-t$" in the above equation. It's easy to see that adding the two versions gives

$$2\langle R_+(0) f, f \rangle = \int_{\partial_- S^*M} \int_{-\infty}^{\infty} f' \circ \varphi_s(z_w) \int_{-\infty}^{\infty} f \circ \varphi_t\big(\varphi_s(z_w)\big) dt\, ds\, |\mu_\partial(w)|$$

$$= \int_{\partial_- S^*M} \left[\int_{-\infty}^{\infty} f' \circ \varphi_s(z_w) ds \right]^2 |\mu_\partial(w)| = \int_{\partial_- S^*M} [If(w)]^2 |\mu_\partial(w)|.$$

On the other hand, we have by (3.11) in the paper that

$$2\langle R_+(0) f, f \rangle = \langle R_+(0) f, f \rangle - \langle f, R_-(0) f \rangle = \langle \Pi f, f \rangle$$

and so

$$\langle \Pi f, f\rangle = \langle If, If\rangle = \langle I^*If, f\rangle.$$

By the polarization identity, this shows that indeed $\Pi = I^*I$ over $C_c^\infty(\partial\_S^*M)$.

10 Page 2887 (PDF Page 32) Bounding the Gradient of the Resolvent

$$|u_+(z)| \leqslant C_k\rho(z)^k\|\widetilde{f}\|_{L^\infty}\int_0^\infty e^{-kt}\mathrm{d}t \leqslant C_k\rho(z)^k\|\widetilde{f}\|_{L^\infty}.$$

To estimate ∇u, it suffices to estimate $\|\overset{v}{\nabla} u\|_G$ and $\|\overset{h}{\nabla} u\|_G$. Using the decomposition of $\mathrm{d}\varphi_t$ in the splitting (2.25) in terms of Jacobi fields [39, Lemma 1.40], we have for each $V \in \mathcal{V}$ of Sasaki norm 1

$$\left|G(\overset{v}{\nabla} u_+(z), V)\right|$$
$$\leqslant C\int_0^\infty \rho^k(\varphi_t(z))\left(\|\widetilde{f}\|_{L^\infty}\left|\frac{\mathrm{d}\rho}{\rho}(\varphi_t(z))\right|_g + \|\nabla\widetilde{f}\|_{L^\infty}\right)(|Y_t(z)|_g + |Y_t'(z)|_g)\mathrm{d}t$$

where $Y_t(z)$ is the Jacobi field solving

$$Y_t''(z) + \mathcal{R}(Y_t(z), \dot{x}(t))\dot{x}(t) = 0, \quad Y_0(z) = 0, \quad Y_0'(z) = V$$

if $\mathcal{R}$ denotes the Riemann curvature tensor of g and $x(t) = \pi(\varphi_t(z))$. Since the sectional curvatures at x are uniformly pinched in $(-1 - c\rho(x), -1 + c\rho(x))$ for some c uniform, and since $\rho(\varphi_t(z)) = \mathcal{O}(e^{-t})$ uniformly in z, we get $\mathcal{R}(Y_t(z), \dot{x}(t))\dot{x}(t) = -Y_t(z) + \mathcal{O}(e^{-t}|Y_t(z)|)$, and by Gronwall's inequality we deduce that there is $C > 0$ so that for each t and each $z \in W_+^\epsilon$

$$|Y_t(z)|_g + |Y_t'(z)|_g \leqslant Ce^t.$$

One has $|\mathrm{d}\rho/\rho|_g = 1$ in the region $\rho \leqslant \epsilon$ and using the uniform estimates (3.15) and (2.11), we deduce that there is $C_k > 0$ such that

$$\|\overset{v}{\nabla} u_+(z)\|_G \leqslant C_k\rho(z)^k(\|\widetilde{f}\|_{L^\infty} + \|\nabla\widetilde{f}\|_{L^\infty}).$$

The same argument works with $|\overset{h}{\nabla} u_+(z)|$.

In this section I'd like to discuss a few of the steps involved here, because they're not easy! First I'd like to show how they bound the following quantity above:

$$\left|G\left(\overset{v}{\nabla} u_+(z), V\right)\right|.$$

Since $\mathbb{R}X$, $\mathcal{H}$, and $\mathcal{V}$ are all perpendicular to eachother with respect to G, we have that (i.e. note the tilde over $\tilde{f}$)

$$G\left(\overset{v}{\nabla} u_+(z), V\right) = G(\nabla u_+, V) = du_+(z).V = d\int_0^\infty \rho^k \circ \varphi_t(z) \cdot \tilde{f} \circ \varphi_t(z) dt\,.V.$$

Due to the eventual exponential decay of $\rho^k \circ \varphi_t(z)$ for large enough "t," we can interchange the "d" with the integral. The last quantity will then become

$$\int_0^\infty \left[k\rho^k \frac{d\rho}{\rho}(d\varphi_t(z).V)\tilde{f}\circ\varphi_t(z) + \rho^k\circ\varphi_t(z)d\tilde{f}(d\varphi_t(z).V)\right]dt.$$

It's not hard to see that this is bounded in absolute value by

$$k\int_0^\infty \rho^k\circ\varphi_t(z)\left(\left|\frac{d\rho}{\rho}\circ\varphi_t(z)\right|_g \left\|\tilde{f}\right\|_{L^\infty} + \|\nabla f\|_{L^\infty}\right)|d\varphi_t(z).V|_g dt$$

Now, how can we bound $|d\varphi_t(z).V|_g$? Looking at equation (7.1) above and using the (nontrivial) facts that the musical isomorphisms and the maps $\widetilde{d\pi}\big|_{\mathbb{R}\tilde{X}\oplus\tilde{\mathcal{H}}}$, and $\tilde{\mathcal{K}}\big|_{\tilde{\mathcal{V}}}$ are isometries with respect to the Sasaki metric, we have that

$$|d\varphi_t(z).V|_g \le \left|J_{z,V}(t)\right| + \left|D_t J_{z,V}(t)\right|.$$

In the paper's notation, $Y_t(z) = J_{z,V}(t)$ and $Y_t' = D_t J_{z,V}(t)$ and so we indeed get that

$$\left|G\left(\overset{v}{\nabla}u_+(z), V\right)\right| \le k\int_0^\infty \rho^k\circ\varphi_t(z)\left(\left|\frac{d\rho}{\rho}\circ\varphi_t(z)\right|_g \left\|\tilde{f}\right\|_{L^\infty} + \|\nabla f\|_{L^\infty}\right)\left(|Y_t(z)|_g + |Y_t'(z)|_g\right)dt.$$

Observe that $|d\rho/\rho|_g \equiv 1$ near ∂M due to the equation in the paper at the top of page 2858.

I don't however understand how they bound $|Y_t(z)|_g + |Y_t'(z)|_g$. The following is the closest that I could up with. Fix any $z \in S^*M$ and observe that by the symmetries of the Riemann curvature tensor we can apply the polarization identity to get that for any $w \in S_{\pi(z)}M$,

$$\langle\mathcal{R}(Y_t,\dot{x})\dot{x},w\rangle = \frac{1}{4}\left[\langle\mathcal{R}(Y_t+w,\dot{x})\dot{x},Y_t+w\rangle - \langle\mathcal{R}(Y_t-w,\dot{x})\dot{x},Y_t-w\rangle\right]$$

Since $V\in\tilde{\mathcal{V}}$, we have that $J_{z,V}$ is a normal Jacobi field along $x(t)$ and hence $Y_t \perp \dot{x}$. So, supposing furthermore that $w\perp\dot{x}$ as well, we get that the above quantity is equal to (here "sec" denotes "sectional curvature")

$$\frac{1}{4}\left[\sec(Y_t+w,\dot{x})\,|Y_t+w|_g^2 - \sec(Y_t-w,\dot{x})\,|Y_t-w|_g^2\right]$$

$$= \frac{1}{4}\left[\left(-1+E(Y_t+w,\dot{x})\right)|Y_t+w|_g^2 - \left(-1+E(Y_t-w,\dot{x})\right)|Y_t-w|_g^2\right]$$

where $E : TM\times TM \to \mathbb{R}$ is some function. Hence we get that

$$\langle\mathcal{R}(Y_t,\dot{x})\dot{x},w\rangle = -\langle Y_t,w\rangle + E(Y_t+w,\dot{x})|Y_t+w|_g^2 - E(Y_t-w,\dot{x})|Y_t-w|_g^2$$

Since E is bounded in absolute value by $c\rho(t) > 0$, we conclude that

$$|\langle \mathcal{R}(Y_t, \dot{x})\dot{x} - (-Y_t), w\rangle| \leq c\rho(t)\left(|Y_t + w|_g^2 + |Y_t - w|_g^2\right) \leq 2cC\rho(z)e^{-t}\left(|Y_t|_g^2 + 1\right).$$

In fact, the inequality of the leftmost and rightmost quantities above extends to all $w \in S_{\pi(z)}M$ since if $w = \dot{x}$, then the leftmost quantity here is zero. So we get that

$$\mathcal{R}\big(Y_t(z), \dot{x}(t)\big)\dot{x}(t) = -Y_t(z) + \mathcal{O}\left(\rho(z)e^{-t}\left(|Y_t(z)|_g^2 + 1\right)\right).$$

Unfortunately, this is not the equation in the paper and it doesn't seem to me that Gronwall's inequality applies to the above equation. So I don't know how to proceed at the moment… Sorry!

11 yPage 2887 (PDF page32) Geodesic vector field on ${}^0S^*\overline{M}$

To prove the last statement, we first notice that $|\partial_z^\alpha \varphi_t(z)| \leqslant C_\alpha e^{c_0|\alpha|\cdot|t|}$ for some c_0 by using Gronwall's inequality and the fact that the vector field X has Lipschitz constants uniformly bounded on the compact manifold ${}^0S^*\overline{M}$. Then take $k > N + c_0|\alpha|$, we have

Let's see why the highlighted statement is true by computing X over ${}^0S^*\overline{M}$. Let us take our standard coordinates (ρ, y, ξ, η) of T^*M and compose them with $(F^*)^{-1}$ to obtain coordinates of ${}^0T^*\overline{M}$ where F^* is the map described in Section 5. Then we get that the coordinates of ${}^0T^*\overline{M}$ and T^*M are related by

$$(11.1) \qquad \bar{\rho} = \rho, \qquad \bar{y} = y, \qquad \bar{\xi}_0 = \rho\xi_0, \qquad \bar{\eta} = \rho\eta$$

where the bars indicate that they are coordinates of ${}^0T^*\overline{M}$. Because of the first two relations, we abuse notation by dropping writing the bars over ρ and y as we did with the b-cotangent bundle. These are in fact the local coordinate expressions for $F^* : T^*\overline{M} \to {}^0T^*\overline{M}$, from which we see that the differential of F^* in these coordinates is given by

$$\begin{bmatrix} \frac{\partial\rho}{\partial\rho} & \frac{\partial\rho}{\partial y^1} & \cdots & \frac{\partial\rho}{\partial y^n} & \frac{\partial\rho}{\partial\xi_0} & \frac{\partial\rho}{\partial\eta_1} & \cdots & \frac{\partial\rho}{\partial\eta_n} \\ \frac{\partial y^1}{\partial\rho} & \frac{\partial y^1}{\partial y^1} & \cdots & \frac{\partial y^1}{\partial y^n} & \frac{\partial y^1}{\partial\xi_0} & \frac{\partial y^1}{\partial\eta_1} & \cdots & \frac{\partial y^1}{\partial\eta_n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial y^n}{\partial\rho} & \frac{\partial y^n}{\partial y^1} & \cdots & \frac{\partial y^n}{\partial y^n} & \frac{\partial y^n}{\partial\xi_0} & \frac{\partial y^n}{\partial\eta_1} & \cdots & \frac{\partial y^n}{\partial\eta_n} \\ \frac{\partial\bar{\xi}_0}{\partial\rho} & \frac{\partial\bar{\xi}_0}{\partial y^1} & \cdots & \frac{\partial\bar{\xi}_0}{\partial y^n} & \frac{\partial\bar{\xi}_0}{\partial\xi_0} & \frac{\partial\bar{\xi}_0}{\partial\eta_1} & \cdots & \frac{\partial\bar{\xi}_0}{\partial\eta_n} \\ \frac{\partial\bar{\eta}_1}{\partial\rho} & \frac{\partial\bar{\eta}_1}{\partial y^1} & \cdots & \frac{\partial\bar{\eta}_1}{\partial y^n} & \frac{\partial\bar{\eta}_1}{\partial\xi_0} & \frac{\partial\bar{\eta}_1}{\partial\eta_1} & \cdots & \frac{\partial\bar{\eta}_1}{\partial\eta_n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial\bar{\eta}_n}{\partial\rho} & \frac{\partial\bar{\eta}_n}{\partial y^1} & \cdots & \frac{\partial\bar{\eta}_n}{\partial y^n} & \frac{\partial\bar{\eta}_n}{\partial\xi_0} & \frac{\partial\bar{\eta}_n}{\partial\eta_1} & \cdots & \frac{\partial\bar{\eta}_n}{\partial\eta_n} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ \xi_0 & 0 & \cdots & 0 & \rho & 0 & \cdots & 0 \\ \eta_1 & 0 & \cdots & 0 & 0 & \rho & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \eta_n & 0 & \cdots & 0 & 0 & 0 & \cdots & \rho \end{bmatrix}$$

So applying this matrix to X as written out in (2.3) in the paper gives us that in ${}^0T^*\overline{M}$, X is equal to

$$X = \rho^2\xi_0\partial_\rho + \rho^2 h_\rho^{ij}\eta_i\partial_{y^j} + \left[\rho^2\xi_0^2 - \rho^2\left(\xi_0^2 + |\eta|_{h_\rho}^2\right) - \frac{1}{2}\rho^3\partial_\rho|\eta|_{h_\rho}^2\right]\partial_{\bar{\xi}_0}$$
$$+ \left[\rho^2\xi_0\eta_k - \frac{1}{2}\rho^3\partial_{y^k}|\eta|_{h_\rho}^2\right]\partial_{\bar{\eta}_k}$$

Canceling the two "$\rho^2\xi_0^2$" terms and using (11.1) above turns this into

$$X = \rho\bar{\xi}_0\partial_\rho + \rho h_\rho^{ij}\bar{\eta}_i\partial_{y^j} - \left[|\bar{\eta}|_{h_\rho}^2 + \frac{1}{2}\rho\partial_\rho|\bar{\eta}|_{h_\rho}^2\right]\partial_{\bar{\xi}_0} + \left[\bar{\xi}_0\bar{\eta}_k - \frac{1}{2}\rho\partial_{y^k}|\bar{\eta}|_{h_\rho}^2\right]\partial_{\bar{\eta}_k}.$$

Notice that this restricts to a vector field on ${}^0S^*\overline{M}$ that's smooth even up to the boundary. This proves the highlighted claim.

12 Page 2888 (PDF page 33) Projection Operator Mapping Property

It is straightforward to see that π_m^* maps continuously

$$\pi_m^* : \rho^{-m}C^\infty(\overline{M}; \otimes_S^m T^*\overline{M}) \to C^\infty(\overline{S^*M}).$$

To see that π_m^* maps between the claimed spaces is simply a calculation. Observe that for any $f \in C^\infty(\overline{M}; \otimes_S^m T^*\overline{M})$ we have in boundary coordinates of $\overline{M}$ described at the top of page 2858

$$\pi_m^* f\left(x, \bar{\xi}_0\frac{d\rho}{\rho} + \eta_\lambda dy^\lambda\right) = f\left(x, \otimes^m\left(\bar{\xi}_0\frac{d\rho}{\rho} + \eta_\lambda dy^\lambda\right)^\#\right) = f\left(x, \otimes^m\left(\rho\bar{\xi}_0 + \rho^2 h^{\mu\nu}\eta_\mu\frac{\partial}{\partial y^\nu}\right)\right)$$

$$= \rho^m f\left(x, \otimes^m \left(\bar{\xi}_0 + \rho h^{\mu\nu}\eta_\mu \frac{\partial}{\partial y^\nu}\right)\right).$$

13 Page 2888 (PDF page 33) The Symmetrized Derivative of a 1-Form

Recall also that for $m = 1$ and f a smooth 1-form, $2Df = \mathcal{L}_{f^\sharp} g$ where $\mathcal{L}$ is the Lie derivative and $f^\sharp$ is the dual vector field to f through g. For a tensor $f = \rho^{-m}\widetilde{f}$ with $\widetilde{f} \in C^\infty(\overline{M}; \otimes_S^m T^*\overline{M})$, one has for $|\xi|_g = 1$

The fact stated in the highlighted text can be seen by a simple calculation. For a smooth 1-form f we have in coordinates (x^i) that

$$\nabla f = (\partial_j f_i - f_r \Gamma_{ji}^r) dx^i \otimes dx^j.$$

and so

$$Df = \frac{1}{2}(\partial_i f_j + \partial_j f_i - 2 f_r \Gamma_{ij}^r) dx^i \otimes dx^j.$$

On the other hand,

$$\mathcal{L}_{f^\#} g = \left(f^\#(g_{ij}) + g_{rj}\partial_i (f^\#)^r + g_{ir}\partial_j (f^\#)^r\right) dx^i \otimes dx^j.$$

Let's furthermore suppose that (x^i) are normal coordinates centered at a point $p \in M$. It's well known that at the center of such coordinates $g_{ij} = \delta_{ij}$ (i.e. the Kronecker delta notation), $\partial_r g^{ij} = 0$, and $\Gamma_{ij}^r = 0$. Hence we have that at p

$$Df = \frac{1}{2}(\partial_i f_j + \partial_j f_i) dx^i \otimes dx^j.$$

Meanwhile at p it holds that $f^\#(g_{ij}) = 0$ and $\partial_i (f^\#)^r = \delta^{rs}\partial_i f_s$ and so at p

$$\mathcal{L}_{f^\#} g = (\delta_{rj}\delta^{rs}\partial_i f_s + \delta_{ir}\delta^{rs}\partial_j f_s) dx^i \otimes dx^j = (\partial_i f_j + \partial_j f_i) dx^i \otimes dx^j.$$

So indeed $2Df = \mathcal{L}_{f^\#} g$ everywhere on M.

14 Page 2889 (PDF page 34) Covariant Derivatives Near the Boundary

and $f_j \in \rho^{-m+1}C^\infty(\overline{M}; \otimes_S^j T^*\partial\overline{M})$. First we recall (see [4, Theorem 1.159]) that for $g = \bar{g}/\rho^2$,

$$\text{(3.21)} \qquad \nabla_X^g Y = \nabla_X^{\bar{g}} Y - \frac{\mathrm{d}\rho}{\rho}(X)Y - \frac{\mathrm{d}\rho}{\rho}(Y)X + \rho^{-1}\bar{g}(X,Y)\partial_\rho.$$

Using this and the Koszul formula, we have

$$\text{(3.22)} \qquad \nabla^g \mathrm{d}\rho = \frac{1}{2}\partial_\rho h_\rho + \frac{\mathrm{d}\rho^2}{\rho} - \frac{h_\rho}{\rho}.$$

If α is a smooth tangential 1-form in a collar $(0,\epsilon)_\rho \times \partial\overline{M}$ near $\partial\overline{M}$ (that is $\iota_{\partial_\rho}\alpha = 0$), we have from the Koszul formula that $(\nabla^g\alpha)(\partial_\rho, \partial_\rho) = 0$ near $\partial\overline{M}$. If in addition α is smooth up to $\partial\overline{M}$, we also get from (3.21)

$$\text{(3.23)} \qquad \nabla^g\alpha = \frac{\mathrm{d}\rho}{\rho} \otimes \alpha + \alpha \otimes \frac{\mathrm{d}\rho}{\rho} + \alpha', \quad \alpha' \in C^\infty(\overline{M}; \otimes^2 T^*\overline{M}).$$

We also have for q_0 a smooth function near $\partial\overline{M}$

$$D(q_0 \mathrm{d}\rho^{m-1})(\partial_\rho, \dots, \partial_\rho) = \partial_\rho q_0 + (m-1)\rho^{-1}q_0.$$

Here I want to discuss how the above highlighted equations are obtained. For later use, recall that in the coordinates near the boundary being discussed here (c.f. top of page 2858 in the paper), the metrics g and $\bar{g}$ have the form (here I list the metrics in two forms: as tensors over the tangent and cotangent bundles respectively [latter sometimes denoted by $g^{\#\#}$ and $\bar{g}^{\#\#}$])

$$\text{(14.1)} \qquad g = \frac{(d\rho)^2 + \left(h_\rho\right)_{\mu\nu} dy^\mu dy^\nu}{\rho^2} \qquad\qquad g = \rho^2\left[\left(\frac{\partial}{\partial\rho}\right)^2 + \left(h_\rho\right)^{\mu\nu}\frac{\partial}{\partial y^\mu}\frac{\partial}{\partial y^\nu}\right]$$

$$\text{(14.2)} \qquad \bar{g} = (d\rho)^2 + \left(h_\rho\right)_{\mu\nu} dy^\mu dy^\nu \qquad\qquad \bar{g} = \left(\frac{\partial}{\partial\rho}\right)^2 + \left(h_\rho\right)^{\mu\nu}\frac{\partial}{\partial y^\mu}\frac{\partial}{\partial y^\nu}$$

Now, (3.21) in the paper follows from the standard fact (see for instance Proposition 7.29 in (Lee, 2018)) that if $\tilde{g} = e^{2f}g$ are two conformally related metrics for some smooth function f, then for any smooth vector fields X and Y,

$$\nabla_X^{\tilde{g}} Y = \nabla_X^g Y + X(f)Y + Y(f)X - \langle X, Y\rangle \operatorname{grad}_g f$$

(this is provable by a simple calculation in coordinates). Hence in the paper, since $g = e^{2(-\ln(\rho))}\bar{g}$, we have that

$$\nabla_X^g Y = \nabla_X^{\bar{g}} Y - \frac{d\rho}{\rho}(X)Y - \frac{d\rho}{\rho}(Y)X + \frac{1}{\rho}\langle X, Y\rangle_{\bar{g}} \operatorname{grad}_{\bar{g}} \rho.$$

Looking at (14.2) above, we see that $\operatorname{grad}_{\bar{g}} \rho = \partial_\rho$. From here we get (3.21) in the paper.

Next let's see how they got (3.22). Unfortunately (3.21) in the paper applies to vector fields and not covector fields. So we'll apply the musical isomorphism "#" to $d\rho$, then apply (3.21), and then apply the musical isomorphism "♭." For distinction, let $\#, \flat$ and $\overline{\#}, \overline{\flat}$ denote the musical isomorphisms of g and $\bar{g}$ respectively and observe that they're related by $\# = \rho^2 \overline{\#}$ and $\flat = \rho^{-2}\overline{\flat}$. Therefore, we observe that for any smooth vector field X

$$\left(\nabla_X^g d\rho\right)^\# = \nabla_X^g (d\rho)^\# = \nabla_X^g\left(\rho^2 \partial_\rho\right)$$
$$= \nabla_X^{\bar{g}}\left(\rho^2\partial_\rho\right) - \frac{d\rho}{\rho}(X)\rho^2\partial_\rho - \frac{d\rho}{\rho}\left(\rho^2\partial_\rho\right)X + \rho^{-1}\bar{g}\left(X, \rho^2\partial_\rho\right)\partial_\rho.$$

From (14.2) we see that $\bar{g}(X, \partial_\rho) = d\rho(X)$ and so the second and fourth term in the last expression cancel. The third term is simply $-\rho X$. Hence we get that

$$\left(\nabla_X^g d\rho\right)^\# = \nabla_X^{\bar{g}}\left(\rho^2\partial_\rho\right) - \rho X.$$

Applying $\flat$ to both sides gives that

$$\nabla_X^g d\rho = \rho^{-2}\nabla_X^{\bar{g}}\left(\rho^2\partial_\rho\right)^{\overline{\flat}} - \rho^{-1}X^{\overline{\flat}} = \rho^{-2}\nabla_X^{\bar{g}}\left(\rho^2 d\rho\right) - \rho^{-1}X^{\overline{\flat}}. \tag{14.3}$$

By the product rule we have that the second to last term here:

$$\nabla_X^{\bar{g}}\left(\rho^2 d\rho\right) = 2\rho\, d\rho(X)d\rho + \rho^2\nabla_X^{\bar{g}} d\rho. \tag{14.4}$$

To evaluate the second term on the right-hand side here, we need to compute some of the Christoffel symbols in these coordinates. *For the rest of this section, let's agree that Greek indices can only take values from* 1 *to* $(n-1)$. Furthermore, let's let $\rho, 1, \dots, n-1$ denote indices representing components in the directions $\partial_\rho, \partial_{y^1}, \dots, \partial_{y^{n-1}}$ respectively. Then, computing the explicit equation for the Christoffel symbols of $\bar{g}$ gives (see also Corollary 6.42 in (Lee, 2018))

$$\bar{\Gamma}_{\rho\rho}^\rho = \bar{\Gamma}_{\rho\alpha}^\rho = \bar{\Gamma}_{\alpha\rho}^\rho = 0, \quad \bar{\Gamma}_{\alpha\beta}^\rho = -\frac{1}{2}\partial_\rho\left(h_\rho\right)_{\alpha\beta}.$$

Plugging this into (14.4) and then the result into (14.3) gives that

$$\nabla_X^g d\rho = 2\rho^{-1}d\rho(X)d\rho - \frac{1}{2}\partial_\rho\left(h_\rho\right)_{\alpha\beta}X^\alpha dy^\beta - \rho^{-1}X^{\overline{\flat}}$$

Observing that $d\rho(X)d\rho - X^{\overline{\flat}}$ is the same things as $\overline{\flat}$ applied to the orthogonal projection of X onto the level set of ρ with respect to $\bar{g}$, we have that the above equation can be rewritten as

$$\nabla_X^g d\rho = \rho^{-1}d\rho(X)d\rho - \frac{1}{2}\partial_\rho\left(h_\rho\right)_{\alpha\beta}X^\alpha dy^\beta - \rho^{-1}\left(X^\top\right)^{\overline{\flat}}$$

where $X^\top$ is that projection of X onto the level set of ρ. It isn't hard to see from (14.2) above that this equation implies that

$$\nabla^g d\rho = \rho^{-1}(d\rho)^2 - \frac{1}{2}\partial_\rho h_\rho - \rho^{-1}h_\rho,$$

which is (3.22) in the paper except that the sign on $(1/2)\partial_\rho h_\rho$ is switched – a typo in the paper I believe.

Now suppose that α is a smooth tangential 1-form, let's show why $(\nabla^g \alpha)\big(\partial_\rho, \partial_\rho\big) = 0$. Notice that by (3.21) in the paper, the left-hand side here is equal to

$$\nabla^g_{\partial_\rho}\alpha\big(\partial_\rho\big) = \langle \nabla^g_{\partial_\rho}(\alpha^\#), \partial_\rho\rangle_g = \langle \nabla^{\bar{g}}_{\partial_\rho}(\alpha^\#) - \frac{d\rho}{\rho}(\alpha^\#)\partial_\rho - \frac{d\rho}{\rho}\big(\partial_\rho\big)\alpha^\# + \rho^{-1}\bar{g}\big(\alpha^\#, \partial_\rho\big), \partial_\rho\rangle_g.$$

From (14.1) and (14.2) it's not hard to see that $d\rho(\alpha^\#) = 0$, $\langle \alpha^\#, \partial_\rho\rangle_g = 0$, and that $\bar{g}\big(\alpha^\#, \partial_\rho\big)$. So we get that the above quantity is really equal to (notice the bar on the $\bar{g}$ here)

$$\rho^{-2}\,\langle \nabla^{\bar{g}}_{\partial_\rho}(\alpha^\#), \partial_\rho\rangle_{\bar{g}}.$$

It's not hard to see that $\alpha^\#$ can be extended to a smooth vector field in a neighborhood of any point that is constantly perpendicular to ∂_ρ. By the product rule, we then get that the above quantity is equal to

$$\rho^{-2}\,\langle \alpha^\#, \nabla^{\bar{g}}_{\partial_\rho}\partial_\rho\rangle_{\bar{g}}.$$

As before, an explicit computation shows that $\bar{\Gamma}^\rho_{\rho\rho} = \bar{\Gamma}^\alpha_{\rho\rho} = 0$, and so this quantity is indeed zero.

Next let's show how they got (3.23). We do this the same way how we demonstrated from where (3.22) in the paper came from. Suppose that α is a tangential 1-form smooth all the way up to ∂M. Then for any smooth vector field X we have by (3.21) in the paper that

$$\nabla^g_X \alpha^\# = \nabla^{\bar{g}}_X \alpha^\# - \frac{d\rho}{\rho}(X)\alpha^\# - \frac{d\rho}{\rho}(\alpha^\#)X + \rho^{-1}\bar{g}(X, \alpha^\#)\partial_\rho.$$

By (14.1) it's straightforward to see that the third term on the right-hand side here is zero. Observe also that $\bar{g}(X, \alpha^\#) = \rho^2\bar{g}\big(X, \alpha^{\bar{\#}}\big) = \rho^2\alpha(X)$. Plugging this into the right-hand side above and then taking $\flat$ if both sides gives that (here I use that $\partial_\rho^\flat = \rho^{-2}d\rho$)

$$\nabla^g_X\alpha = \frac{1}{\rho^2}\nabla^{\bar{g}}_X(\rho^2\alpha) - \frac{d\rho}{\rho}(X)\alpha + \rho^{-1}\big(\rho^2\alpha(X)\big)\rho^{-2}d\rho$$

$$= \frac{2}{\rho}d\rho(X)\alpha + \nabla^{\bar{g}}_X\alpha - \frac{d\rho}{\rho}(X)\alpha + \alpha(X)\frac{d\rho}{\rho} = \nabla^{\bar{g}}_X\alpha + \frac{d\rho}{\rho}(X)\alpha + \alpha(X)\frac{d\rho}{\rho}.$$

It's not hard to see that this implies that

$$\nabla^g\alpha = \nabla^{\bar{g}}\alpha + \alpha \otimes \frac{d\rho}{\rho} + \frac{d\rho}{\rho} \otimes \alpha.$$

Since $\nabla^{\bar{g}}\alpha \in C^\infty(\bar{M}; \otimes^2\, T^*\bar{M})$, this proves (3.23).

Lastly, let q_0 be a smooth function near $\partial\bar{M}$. Then we have that

$$D(q_0 d\rho^{m-1})(\partial_\rho, \dots, \partial_\rho) = \nabla^g(q_0 d\rho^{m-1})(\partial_\rho, \dots, \partial_\rho)$$

$$= \partial_\rho q_0 d\rho^{m-1}(\partial_\rho, \dots, \partial_\rho) + q_0 \sum_{j=1}^{m-1} \underbrace{d\rho \otimes \dots \otimes \nabla^g_{\partial_\rho}}_{j}(d\rho) \otimes \dots \otimes d\rho(\partial_\rho, \dots, \partial_\rho).$$

Now, using (3.22) in the paper and the facts that $d\rho(\partial_\rho) = 1$, $\partial_\rho h_\rho(\partial_\rho) = 0$, and $h_\rho(\partial_\rho) = 0$ we get that this quantity is indeed equal to

$$\partial_\rho q_0 + q_0(m-1)\rho^{-1}.$$

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we get for each $u \in C^\infty(S^*M)$

$$\begin{aligned}
&\|\overset{v}{\nabla} Xu\|^2_{L^2} - \|X\overset{v}{\nabla} u\|^2_{L^2} \\
&\quad = \langle \overset{v}{\operatorname{div}} \overset{v}{\nabla} Xu, Xu\rangle_{L^2} + \langle X^2 \overset{v}{\nabla} u, \overset{v}{\nabla} u\rangle_{L^2} - \int_{\partial W_\epsilon} \langle \overset{v}{\nabla} u, X\overset{v}{\nabla} u\rangle \mu_\epsilon \\
&\quad = \langle (\overset{v}{\operatorname{div}} X^2 \overset{v}{\nabla} - X\overset{v}{\operatorname{div}} \overset{v}{\nabla} X)u, u\rangle_{L^2} - \int_{\partial W_\epsilon} \langle \overset{v}{\nabla} u, (X\overset{v}{\nabla} - \overset{v}{\nabla} X)u\rangle \mu_\epsilon \\
&\quad = \langle (\overset{h}{\operatorname{div}} \overset{v}{\nabla} X - \overset{v}{\operatorname{div}} X \overset{h}{\nabla})u, u\rangle_{L^2} + \int_{\partial W_\epsilon} \langle \overset{v}{\nabla} u, \overset{h}{\nabla} u\rangle \mu_\epsilon \\
&\quad = \langle (\overset{h}{\operatorname{div}} \overset{v}{\nabla} X - \overset{v}{\operatorname{div}} \overset{h}{\nabla} X - \overset{v}{\operatorname{div}} \mathcal{R} \overset{v}{\nabla})u, u\rangle_{L^2} + \int_{\partial W_\epsilon} \langle \overset{v}{\nabla} u, \overset{h}{\nabla} u\rangle \mu_\epsilon \\
&\quad = -\langle (nX^2 + \overset{v}{\operatorname{div}} \mathcal{R} \overset{v}{\nabla})u, u\rangle_{L^2} + \int_{\partial W_\epsilon} \langle \overset{v}{\nabla} u, \overset{h}{\nabla} u\rangle \mu_\epsilon
\end{aligned}$$

Here I'd like to explain how the authors got the highlighted step above. For the integral over ∂W_ε they simply used the first identity in (3.20) in the paper. Now let's see what they did with the $\overset{v}{\operatorname{div}} X^2 \overset{v}{\nabla}$ operator. The following identities come from Lemma 2.1 in the reference "[41]" mentioned in the paper:

(15.1) $$\left[X, \overset{v}{\operatorname{div}}\right] = -\overset{h}{\operatorname{div}},$$

(15.2) $$\left[X, \overset{h}{\operatorname{div}}\right] = -\overset{v}{\operatorname{div}} R.$$

These are simply obtained by taking the adjoint of the first and second identities in (3.20) in the paper. Hence, we have that

$$\overset{v}{div}\, X^2 \overset{v}{\nabla}$$

The operator of interest

$= X \overset{v}{div}\, X\overset{v}{\nabla} + \overset{h}{div}\, X\overset{v}{\nabla}$	Used (15.1) above.
$= X \overset{v}{div}\, X\overset{v}{\nabla} + \overset{h}{div}\,\overset{v}{\nabla}X - \overset{h}{div}\,\overset{h}{\nabla}$	Used first identity in (3.20) in the paper on the second term.
$= X \overset{v}{div}\,\overset{v}{\nabla}X - X\overset{v}{div}\,\overset{h}{\nabla} + \overset{h}{div}\,\overset{v}{\nabla}X - \overset{h}{div}\,\overset{h}{\nabla}$	Used first identity in (3.20) in the paper on the first term.
$= X \overset{v}{div}\,\overset{v}{\nabla}X - \overset{v}{div}\, X\overset{h}{\nabla} + \overset{h}{div}\,\overset{h}{\nabla} + \overset{h}{div}\,\overset{v}{\nabla}X - \overset{h}{div}\,\overset{h}{\nabla}$	Used (15.1) above on the second term.
$= X \overset{v}{div}\,\overset{v}{\nabla}X - \overset{v}{div}\, X\overset{h}{\nabla} + \overset{h}{div}\,\overset{v}{\nabla}X$	Two of the terms canceled out (namely the horizontal Laplacians).

Plugging this into the highlighted equations, it should be clear from here how the highlighted step follows (in particular, notice that the the operator $X\,\overset{v}{\mathrm{div}}\,\overset{v}{\nabla}X$ cancels out).

16 References

The following are references used in addition to the paper being analyzed in this document:

Lee, J. M. (2018). *Introduction to Riemannian Manifolds* (2nd ed.). Cham, Switzerland: Springer International Publishing AG.

Paternain, G., Salo, M., & Uhlmann, G. (2022). *Geometric Inverse Problems, With Emphasis in Two Dimensions.* Cambridge: Cambridge University Press & Assessment.