Haim's Notes About

The X-Ray Transform for Connections in Negative Curvature

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1 Table of Contents

2 Notations and Conventions

Convention 2.1: For any smooth manifold M possibly with boundary, I let $\mathfrak{X}(M)$ and $\mathfrak{X}^*(M)$ denote the space of all smooth vector fields and smooth covector fields over M respectively.

Similarly, if $\mathcal{E}$ is any vector bundle over M, I let $\Gamma(\mathcal{E})$ and $\Gamma(\mathcal{E}^*)$ denote the space of all smooth sections of $\mathcal{E}$ and its dual bundle $\mathcal{E}^*$ respectively.

Convention 2.2: I use the Einstein summation convention here.

3 Page 93 (PDF page 11) Total Covariant Derivative over Bundle $\mathcal{E}$

which satisfies $\nabla_{fY}u = f\nabla_Y u$ and $\nabla_Y(fu) = (Yf)u + f\nabla_Y u$ for $f \in C^\infty(M)$ and $u \in C^\infty(M;\mathcal{E})$. There is a corresponding map

$$\nabla : C^\infty(M;\mathcal{E}) \to C^\infty(M; T^*M \otimes \mathcal{E}).$$

My understanding is that for any smooth section $u : M \to \mathcal{E}$, ∇u denotes the tensor $\nabla u : \mathfrak{X}(M) \times \Gamma(\mathcal{E}^*) \to \mathbb{C}$ given by

$$\nabla u(X,\Omega) = \Omega(\nabla_X u)$$

Suppose that $(e_1, \dots, e_n)$ is a frame of $\mathcal{E}$ over U as in this paragraph in the paper (I drop the assumption that it's orthonormal because I don't believe it's needed for what's stated below). Suppose also that (x^i) are local coordinates over U and let Γ_{ij}^k be the coefficients in

$$\nabla_{\partial_i} e_j = \Gamma_{ij}^k e_k.$$

Then we have that for any $v \in T_x M$ and any $\omega \in \mathcal{E}_x^*$

$$\nabla u(v,\omega) = \omega_r e^r \left(\left[v^i \frac{\partial u^k}{\partial x^i} + v^i u^j \Gamma_{ij}^k \right] e_k \right) = \left(\frac{\partial u^k}{\partial x^i} + u^j \Gamma_{ij}^k \right) v^i \omega_k$$

Hence over U,

$$\nabla u = \left(\frac{\partial u^k}{\partial x^i} + u^j \Gamma_{ij}^k \right) dx^i \otimes e_k = \left(d(u^k) + u^j \Gamma_{ij}^k dx^i \right) \otimes e_k,$$

$$\nabla_Y u = \left(Y^i \frac{\partial u^k}{\partial x^i} + u^j Y^j \Gamma_{ij}^k \right) e_k = \left(d(u^k) + u^j \Gamma_{ij}^k dx^i \right)(Y) e_k.$$

This is the same as equations given in the paper where they set $A_j^k = \Gamma_{ij}^k dx^i$.

4 Page 93 (PDF page 11) Alternating Tensors Tensored with $\mathcal{E}$

If ∇ is a connection on a complex vector bundle $\mathcal{E}$, we can define a linear operator

$$\nabla : C^\infty(M; \Lambda^k(T^*M) \otimes \mathcal{E}) \to C^\infty(M; \Lambda^{k+1}(T^*M) \otimes \mathcal{E})$$

I believe that what they mean here with the notation $C^\infty(M; \Lambda^k(T^*M) \otimes \mathcal{E})$ doesn't align with standard usage since $\Lambda^k(T^*M)$ is not a tensor product. Here is my interpretation of what this notation means is (here "$\mathfrak{X}(M)$" is being multiplied k times):

$$\begin{aligned} &C^\infty(M;\Lambda^k(T^*M)\otimes\mathcal{E}) \\ &= \{\text{multilinear and smooth } u : \mathfrak{X}(M)\times\ldots\times\mathfrak{X}(M)\times\mathcal{E}^*\to\mathbb{C} \\ &: u\big(v_{\sigma(1)},\ldots,v_{\sigma(1)},l\big) = (\operatorname{sgn}\sigma)u(v_1,\ldots,v_k,l)\ \forall\sigma\in S_k\big\}. \end{aligned}$$

In other words, $C^\infty(M;\Lambda^k(T^*M)\otimes\mathcal{E})$ is the subspace of tensors $C^\infty(M;T^*M\otimes\ldots\otimes T^*M\otimes\mathcal{E})$ that are alternating in the first k arguments.

5 Page 93 (PDF page 11) Wedge Product of Alternating tensor with Alternating Tensors Tensored with $\mathcal{E}$

for $k\geq 1$ by requiring that

$$\nabla(\omega\wedge u) = d\omega\wedge u + (-1)^k\omega\wedge\nabla u, \qquad \omega\in C^\infty(M;\Lambda^k(T^*M)), \quad u\in C^\infty(M;\mathcal{E})$$

To understand this equation, we first need to define the wedge product of a $\bigotimes_{i=1}^k T^*M$ valued tensor and a $\left[\bigotimes_{i=1}^j T^*M\right]\otimes\mathcal{E}$ valued tensor because that's not classically defined since the last argument of the latter type tensor does not live in the space of the arguments of the former. Here's the definition: if $\omega\in\bigotimes_{i=1}^k T^*M$ and $\eta\in\left[\bigotimes_{i=1}^j T^*M\right]\otimes\mathcal{E}$ are tensors, then we define

$$\omega\wedge\eta\big(v_1,\ldots,v_{k+j},l\big) = \frac{1}{k!\,j!}\sum_{\sigma\in S_{k+j}}(\operatorname{sgn}\sigma)\omega\big(v_{\sigma(1)},\ldots,v_{\sigma(k)}\big)\eta\big(v_{\sigma(k+1)},\ldots,v_{\sigma(k+j)},l\big)$$

In other words, you perform the tensor product and "alt" operator on the first $k+j$ arguments. It's is easy to see that if $\omega\in\bigotimes_{i=1}^k T^*M, \tilde{\omega}\in\bigotimes_{i=1}^j T^*M$, and $e\in\mathcal{E}$, then

$$\omega\wedge(\tilde{\omega}\otimes e) = (\omega\wedge\tilde{\omega})\otimes e.$$

So for instance, this formula in the paper that defines ∇ on $C^\infty(M;\Lambda^k(T^*M)\otimes\mathcal{E})$ can instead be written as

$$(5.1)\qquad \nabla(\omega\otimes u) = d\omega\otimes u + (-1)^k\omega\wedge\nabla u \qquad \forall\omega\in C^\infty\big(M;\Lambda^k(T^*M)\big),\ \forall u\in C^\infty(M;\mathcal{E})$$

One needs to check that such an operator "∇" exists and is well defined. Let's discuss how this is done. It's now hard to see that the above equation implies that the operator is unique. Next, fix a coordinate chart $(U,(x^i))$ of M (i.e. $U\subseteq M$ is the domain of the chart) and let (r_j) be a frame of $\mathcal{E}$ over U. Any element in $C^\infty(M;\Lambda^k(T^*M)\otimes\mathcal{E})$ can be written as

$$\sum_I{}' \omega_I^j dx^I\otimes r_j$$

where $I=(i_1,\ldots,i_k)$ denotes k-tuple of indices, $dx^I = dx^{i_1}\wedge\ldots\wedge dx^{i_k}$, the prime " $'$ " over the sum means that we only sum over k-tuples I of strictly increasing order (i.e. $i_1<\cdots<i_k$), and $\omega_{I,j}$ are smooth functions over M. Over U we define ∇ to be the operator (linear over $\mathbb{C}$)

$$\nabla\left(\sum_I{}' \omega_I^j dx^I \otimes r_j\right) = \sum_I{}' d\left(\omega_I^j dx^I\right) \otimes r_j + (-1)^k \sum_I{}' \omega_I^j dx^I \wedge \nabla r_j.$$

It's not hard to see that if we show that this operator satisfies (5.1) above over U, then ∇ is well defined over all of M and satisfies (5.1) over all of M. So let's show that (5.1) holds over U. Take any smooth $\omega \in C^\infty(M; \Lambda^k(T^*M))$ and any $u \in C^\infty(M; \mathcal{E})$. Write

$$\omega = \sum_I{}' \omega_I dx^I \quad \text{and} \quad u = u^j r_j.$$

Over U we have that

$$\nabla(\omega \otimes u) = \nabla\left(\sum_I{}' \omega_I u^j dx^I \otimes r_j\right) = \sum_I{}' d\left(\omega_I u^j dx^I\right) \otimes r_j + (-1)^k \sum_I{}' \omega_I u^j dx^I \wedge \nabla r_j$$

$$= \sum_I{}' \left(u^j d(\omega_I dx^I) + du^j \wedge (\omega_I dx^I)\right) \otimes r_j + (-1)^k \sum_I{}' (\omega_I dx^I) \wedge \left(u^j \nabla r_j\right)$$

$$\sum_I{}' d(\omega_I dx^I) \otimes \left(u^j r_j\right) + (-1)^k \sum_I{}' (\omega_I dx^I) \wedge \left(du^j \otimes r_j + u^j \nabla r_j\right).$$

Since $\nabla(u^j r_j) = du^j \otimes r_j + u^j \nabla r_j$, we have that this is indeed equal to $d\omega \otimes u + (-1)^k \omega \wedge \nabla u$.

For reference, I include the formula for the derivative of a $C^\infty(M; T^*M \otimes \mathcal{E})$ tensor. Suppose that $(U, (x^i))$ are local coordinates of M and that (r_i) are a frame of $\mathcal{E}$ over U. Then for any $u \in C^\infty(M; T^*M \otimes \mathcal{E})$ we have over U that

$$\nabla u = \nabla\left(u_i^j dx^i \otimes r_j\right) = \frac{\partial u_i^j}{\partial x^\mu}\left(dx^\mu \wedge dx^i\right) \otimes r_j - u_i^j dx^i \wedge \left(\Gamma_{\mu j}^k dx^\mu \otimes r_k\right),$$

and so we have either the following two convenient forms for ∇u:

$$\nabla u = \frac{\partial u_i^j}{\partial x^\mu}\left(dx^\mu \wedge dx^i\right) \otimes r_j - u_i^j \Gamma_{\mu j}^k \left(dx^i \wedge dx^\mu\right) \otimes r_k$$

$$\nabla u = \left(\frac{\partial u_i^j}{\partial x^\mu} - u_\mu^k \Gamma_{ik}^j\right)\left(dx^\mu \wedge dx^i\right) \otimes r_j$$

(I switched $\mu \leftrightarrow i$ and $j \leftrightarrow k$ labels in the second term in the very last equation).

6 Page 94 (PDF page 12) Curvature of a Connection

then

$$f^{\mathcal{E}}(\sum_{k=1}^{n} u^k e_k) = \sum_{k,l=1}^{n} (dA_l^k + A_m^k \wedge A_l^m)u^l \otimes e_k.$$

Locally one writes

$$f^{\mathcal{E}} = dA + A \wedge A.$$

Let me write this equation in the paper out a bit more explicitly so that it's clear what it means. Suppose that $(U,(x^i))$ are local coordinate of M and that (r_i) are a smooth frame of $\mathcal{E}$ over U. Then for any $u \in C^\infty(M;\mathcal{E})$ over U one can check that $f^{\mathcal{E}}(u)$ can be written in any one of the following forms:

$$\begin{aligned} f^{\mathcal{E}}(u) = \nabla^2 u &= u^\lambda \left(\frac{\partial \Gamma_{i\lambda}^j}{\partial x^\mu} - \Gamma_{\mu\lambda}^m \Gamma_{im}^j\right)\left(dx^\mu \wedge dx^i\right) \otimes r_j \\ &= u^\lambda \left(\frac{\partial \Gamma_{i\lambda}^j}{\partial x^\mu} - \Gamma_{\mu\lambda}^m \Gamma_{im}^j - \frac{\partial \Gamma_{\mu\lambda}^j}{\partial x^i} + \Gamma_{i\lambda}^m \Gamma_{\mu m}^j\right) dx^\mu \otimes dx^i \otimes r_j \\ &= u^\lambda \left(d\left(\Gamma_{i\lambda}^j dx^i\right) + \left(\Gamma_{im}^j dx^i\right) \wedge \left(\Gamma_{\mu\lambda}^m dx^\mu\right)\right) \otimes r_j \\ &= u^\lambda \left(dA_\lambda^j + A_m^j \wedge A_\lambda^m\right) \otimes r_j \end{aligned}$$

where recall the matrix of one forms $\left[A_\lambda^j\right]_{j,\lambda=1}^n = \left[\Gamma_{i\lambda}^j dx^i\right]_{j,\lambda=1}^n$.

It's interesting to note that from the second equation above we can see that if it so happens that $\mathcal{E} = TM$, then for any $V, X, Y \in \mathfrak{X}^*(M)$ and any $\Omega \in \mathfrak{X}^*(M)$,

$$f^{\mathcal{E}}(V)(X,Y,\Omega) = R(X,Y,V,\Omega)$$

where R is the ordinary (1,3)-curvature endomorphism. That's the reason we call $f^{\mathcal{E}}$ the curvature of a general connection $(\mathcal{E},\nabla)$.

In the paragraph before, the authors write

$$f \in C^\infty(M;\Lambda^2(T^*M) \otimes \mathrm{End}(\mathcal{E})) \tag{6.1}$$

Here is my understanding of this notation. We have that $f^{\mathcal{E}} : \Gamma(\mathcal{E}) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \times \Gamma(\mathcal{E}^*) \to \mathbb{C}$ maps

$$f^{\mathcal{E}}(u,X,Y,L) = f^{\mathcal{E}}(u)(X,Y,L)$$

We can naturally associate this with a map of the form $f^{\mathcal{E}} : \Gamma(\mathcal{E}) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to \Gamma(\mathcal{E})$ that maps as follows: if (r_i) is a smooth local frame for $\mathcal{E}$ over an open set U and $\left(l^i\right)$ is its dual coframe, then

$$f^{\mathcal{E}}(u,X,Y) = f^{\mathcal{E}}(u)\left(X,Y,l^i\right)r_i.$$

Considering the bundle of endomorphisms over M : $\mathrm{End}(\mathcal{E})$, then $f^{\mathcal{E}}$ further generates a smooth map of the form $f^{\mathcal{E}}: \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathrm{End}(\mathcal{E})$ given by the following at any $p \in U$:

$$f^{\mathcal{E}}(X,Y)_p(e) = f^{\mathcal{E}}(e)\left(X_p, Y_p, l^i\big|_p\right) r_i\Big|_p$$

This is the closest interpretation that I can get of what them mean by (6.1) above. In fact, it's not hard to check that if $\mathcal{E}$ and ∇ are unitary, then all of the "$\mathrm{End}(\mathcal{E})$" here can be changed to "$\mathrm{End}_{\mathrm{sk}}(\mathcal{E})$" (see the end of page 93 [PDF page 11] of the paper).

7 Page 96 (PDF page 14) Splitting of a covariant derivative of a section of $\mathcal{E}$

> If $u \in C^\infty(SM;\mathcal{E})$, then $\nabla^{\mathcal{E}} u \in C^\infty(SM; T^*(SM) \otimes \mathcal{E})$, and using the Sasaki metric on $T(SM)$ we can identify this with an element of $C^\infty(SM; T(SM) \otimes \mathcal{E})$, and thus we can split according to (3.1)
>
> $$\nabla^{\mathcal{E}} u = (\mathbb{X}u, \overset{\mathrm{h}}{\nabla}{}^{\mathcal{E}} u, \overset{\mathrm{v}}{\nabla}{}^{\mathcal{E}} u), \quad \mathbb{X}u := \nabla^{\mathcal{E}}_X u$$

Here I'd like to discuss the highlighted splitting: in particular what I believe they mean by $\mathbb{X}u$, $\overset{\mathrm{h}}{\nabla}{}^{\mathcal{E}} u$, and $\overset{\mathrm{v}}{\nabla}{}^{\mathcal{E}} u$. Let's first discuss $\overset{\mathrm{h}}{\nabla}{}^{\mathcal{E}} u$. Take any $u \in C^\infty(SM;\mathcal{E})$. They define $\overset{\mathrm{h}}{\nabla}{}^{\mathcal{E}} u$ as follows. First compute $\nabla^{\mathcal{E}} u \in C^\infty(SM; T^*(SM) \otimes \mathcal{E})$ as defined earlier in the paper. Then, raise the index of $\nabla^{\mathcal{E}} u$ in the first argument with respect to the (Sasaki) metric on SM to get $(\nabla^{\mathcal{E}} u)^\sharp \in C^\infty(SM; T(SM) \otimes \mathcal{E})$. Then send this resulting tensor through the unique map

$$\Pi_{\mathcal{H}} : C^\infty(SM; T(SM) \otimes \mathcal{E}) \to C^\infty(SM; \mathcal{H} \otimes \mathcal{E})$$

that satisfies

$$\Pi_{\mathcal{H}}(V \otimes e) = (\pi_{\mathcal{H}} V) \otimes e$$

where $\pi_{\mathcal{H}} V$ is the projection of V onto $\mathcal{H}$ in the orthogonal decomposition $T(SM) = \mathbb{R}X \oplus \mathcal{H} \oplus \mathcal{V}$. We leave it to the reader to check that such a map $\Pi_{\mathcal{H}}$ indeed exists. The result is what we call "$\overset{\mathrm{h}}{\nabla}{}^{\mathcal{E}} u$."

The tensor $\overset{\mathrm{v}}{\nabla}{}^{\mathcal{E}} u$ is defined the same way except that you use maps $\Pi_{\mathcal{V}}$ and $\pi_{\mathcal{V}}$ that are defined analogously. For $\mathbb{X}u$, you almost do the same thing. Instead of using the projection maps $\Pi_{\mathbb{R}X}$ and $\pi_{\mathbb{R}X}$ at the end, you use maps $\mathrm{comp}_X : C^\infty(SM; T(SM) \otimes \mathcal{E}) \to C^\infty(SM; (\mathbb{R}X) \otimes \mathcal{E})$ and $\mathrm{comp}_X : T(SM) \to \mathbb{R}X$ that give the component of your object in the direction of the subspace $\mathbb{R}X$. However, it's simpler to just use the equivalent definition

$$\mathbb{X}u \coloneqq \nabla^{\mathcal{E}}_X u.$$

8 Page 96 (PDF page 14) $\mathbb{X}$ acting on smooth sections of $N \otimes \mathcal{E}$

and we can view $\overset{\mathrm{h}}{\nabla}{}^{\mathcal{E}} u$ and $\overset{\mathrm{v}}{\nabla}{}^{\mathcal{E}} u$ as elements in $C^\infty(SM; N \otimes \mathcal{E})$. The operator $\mathbb{X}$ acts on $C^\infty(SM; \mathcal{E})$ and we can also define a similar operator, still denoted by $\mathbb{X}$, on $C^\infty(SM; N \otimes \mathcal{E})$ by

$$\mathbb{X}(Z \otimes e) := (XZ) \otimes e + Z \otimes (\mathbb{X}e), \quad Z \otimes e \in C^\infty(SM; N \otimes \mathcal{E}) \tag{3.3}$$

In the paper they claim that there exists a unique operator $\mathbb{X}$ acting on $C^\infty(SM; N \otimes \mathcal{E})$ satisfying the highlighted equation. It's not hard to see uniqueness, so let me show existence. Let (b_i) and (r_j) be frames for N and $\mathcal{E}$ respectively over an open set $U \subseteq SM$. Any smooth section of $N \otimes \mathcal{E}$ can be written as $A^{ij} b_i \otimes r_j$. Over U we define $\mathbb{X}$ to be

$$\text{(8.1)} \qquad \mathbb{X}(A^{ij} b_i \otimes r_j) = [X(A^{ij} b_i)] \otimes r_j + (A^{ij} b_i) \otimes (\mathbb{X} r_j).$$

It's not hard to see that if we show that this definition of $\mathbb{X}$ satisfies the above highlighted equation, then this $\mathbb{X}$ is well defined over U (i.e. independent of the (b_i) and (r_j) that we choose) and hence over M as well. Thus, take any $Z \in C^\infty(SM; N)$ and any $e \in C^\infty(SM; \mathcal{E})$ which we write component wise as $Z = Z^i b_i$ and $e = e^j r_j$. Then by (8.1)

$$\mathbb{X}(Z \otimes e) = [X(Z^i e^j b_i)] \otimes r_j + (Z^j e^j b_i) \otimes (\mathbb{X} r_j)$$
$$= [X(e^j) Z^i b_i + e^j X(Z^i b_i)] \otimes r_j + (Z^j b_i) \otimes (e^j \mathbb{X} r_j)$$
$$= [X(Z^i b_i)] \otimes (e^j r_j) + (Z^i b_i) \otimes [X(e^j) r_j + e^j \mathbb{X} r_j].$$
$$= [X(Z)] \otimes e + Z \otimes [\mathbb{X}(e)].$$

9 Page 97 (PDF page 14) The operator $F^{\mathcal{E}}$

if $(x, v) \in SM$, $w \in \{v\}^\perp$, and $e \in \mathcal{E}_{(x,v)}$. The curvature of the connection $\nabla^{\mathcal{E}}$ on $\mathcal{E}$ is denoted $f^{\mathcal{E}} \in C^\infty(M; \Lambda^2 T^* M \otimes \mathrm{End}_{\mathrm{sk}}(\mathcal{E}))$ and it is a 2-form with values in skew-Hermitian endomorphisms of $\mathcal{E}$. In particular, to $f^{\mathcal{E}}$ we can associate an operator $F^{\mathcal{E}} \in C^\infty(SM; N \otimes \mathrm{End}_{\mathrm{sk}}(\mathcal{E}))$ defined by

$$\langle f_x^{\mathcal{E}}(v, w)e, e' \rangle_{\mathcal{E}} = \langle F^{\mathcal{E}}(x, v)e, w \otimes e' \rangle_{N \otimes \mathcal{E}}, \tag{3.5}$$

Here I'd like to explain what this operator $F^{\mathcal{E}}$ is. Let N^* and $\mathcal{E}^*$ denote the dual bundle to N and $\mathcal{E}$ repsectively. Then, $F^{\mathcal{E}}$ is the element of $C^\infty(SM; N \otimes \mathcal{E}^* \otimes \mathcal{E})$ given by

$$F^{\mathcal{E}}(x, v)(\omega, e, l) = f_x^{\mathcal{E}}(e, v, \omega^\sharp, l)$$

where $(x, v) \in SM$, $\omega \in N^*_{(x,v)}$, $e \in \mathcal{E}_x$, $l \in \mathcal{E}^*_x$, and $\omega^\sharp$ denotes the musical isomorphism of g applied to ω in $T_x M$ that raises its indices. Let's show that this definition of $F^{\mathcal{E}}$ satisfies the above highlighted equation. Let (b_i) be an orthonormal basis of N_x and (r_j) be an orthonormal basis of $\mathcal{E}_x$. Let (β^i) and (ρ^j) be dual bases for (b_i) and (r_j) respectively. Now, take any

$(x, v) \in SM$ and any $e \in \mathcal{E}_x$. The quantity $F^{\mathcal{E}}(x, v)e$ simply denotes the tensor that maps $(\omega, l) \mapsto F^{\mathcal{E}}(x, v)(\omega, e, l)$. It's not hard to see that this is explicitly given by

$$F^{\mathcal{E}}(x, v)e = F^{\mathcal{E}}(x, v)(\beta^i, e, \rho^j)b_i \otimes r_j = f_x^{\mathcal{E}}\left(e, v, (\beta^i)^\sharp, \rho^j\right) b_i \otimes r_j.$$

Since (b_i) is orthonormal, it's not hard to see that $(\beta^i)^\sharp = b_i$. Hence we get that (for the rest of this section I will use the Einstein summation convention improperly by not requiring that one index be lower and the other higher)

$$F^{\mathcal{E}}(x, v)e = f_x^{\mathcal{E}}(e, v, b_i, \rho^j)b_i \otimes r_j.$$

Now, take any $w \in N_x$ and $e' \in \mathcal{E}_x$ that we write component wise as $w = w^i b_i$ and $e = e^j r_j$. Then we get that (here we again use that (b_i) and (r_j) are orthonormal)

$$\langle F^{\mathcal{E}}(x, v)e, w \otimes e' \rangle_{N \otimes \mathcal{E}} = \langle f_x^{\mathcal{E}}(e, v, b_i, \rho^j)b_i \otimes r_j, w^i e'^j b_i \otimes r_j \rangle_{N \otimes \mathcal{E}} = f_x^{\mathcal{E}}(e, v, b_i, \rho^j)w^i e'^j.$$

$$= f_x^{\mathcal{E}}(e, v, w, \rho^j)e'^j = \langle f_x^{\mathcal{E}}(e, v, w, \rho^j)r_j, e'^j r_j \rangle_{\mathcal{E}} = \langle f_x^{\mathcal{E}}(v, w)e, e' \rangle_{\mathcal{E}}.$$

Hence the highlighted equation indeed holds.

10 Page 99 (PDF page 17) Vertical and $\mathcal{E}$ Laplacians over SM

We write $\Omega_m = H_m(SM; \mathcal{E}) \cap C^\infty(SM; \mathcal{E})$, and write $\Delta^{\mathcal{E}} = -\overset{\mathrm{v}}{\mathrm{div}}{}^{\mathcal{E}} \overset{\mathrm{v}}{\nabla}{}^{\mathcal{E}}$ for the vertical Laplacian. Notice that since $(\mathcal{E}, \nabla^{\mathcal{E}})$ are pulled back from M to SM, we have in a local orthonormal frame $(e_1, \ldots, e_n)$ the representation

$$\Delta^{\mathcal{E}}\left(\sum_{k=1}^n u^k e_k\right) = \sum_{k=1}^n (\Delta u^k) e_k$$

where $\Delta := -\overset{\mathrm{v}}{\mathrm{div}} \overset{\mathrm{v}}{\nabla}$ is the vertical Laplacian on functions defined in [PSU14c, Section 3].

In his note I'd like to discuss why the highlighted equation holds. In fact, I drop the assumption that $(e_1, \ldots, e_n)$ is orthonormal because as far as I understand it's not needed. Let $\pi : TM \to M$ denote the natural projection and let $\nabla^{\mathcal{E}}$ and $\pi^* \nabla^{\mathcal{E}}$ denote the Hermitian connections of interest. The authors omit writing π^* in their notation, but for purposes of clarity we won't omit it in this section. In this vein, notice that since the e_k's are frames of $\mathcal{E}$ over an open set in M the highlighted equation should really read (I use the Einstein summation convention here so I drop the "Σ")

$$\Delta^{\mathcal{E}}(u^k \pi^* e_k) = (\Delta u^k) \pi^* e_k.$$

We'll prove this equation by first demonstrating that

$$\overset{\mathrm{v}}{\nabla}{}^{\mathcal{E}}(u^k \pi^* e_k) = \left(\overset{\mathrm{v}}{\nabla} u^k\right) \otimes \pi^* e_k, \tag{10.1}$$

(this equation was mentioned in the paper on page 98) To see why this holds, take any $W \in TSM$ and observe that over any coordinates (x^i) contained in the domain of the frame (e_k), the definition of the pullback connection (c.f. my "Miscellaneous notes") tells us that

$$(\pi^*\nabla^{\mathcal{E}})_W(u^k\pi^*e_k) = W(u^k)\pi^*e_k + u^k\pi^*\nabla^{\mathcal{E}}_{d\pi(W)}(e_k) = W(u^k)\pi^*e_k + u^k\pi^*\left(\Gamma^j_{ik}W^ie_j\right)$$

where W^i denote the components of $d\pi(W)$ with respect to $\left(\partial/\partial x^i\right)$ and Γ^j_{ik} are the connection symbols in the equations $\nabla^{\mathcal{E}}_{\partial/\partial x^i}e_k = \Gamma^j_{ik}e_j$. It's not hard to see that this implies that

$$(\pi^*\nabla^{\mathcal{E}})(u^k\pi^*e_k) = du^k \otimes \pi^*e_k + u^k\Gamma^j_{ik}\left(\pi^*dx^i \otimes \pi^*e_j\right).$$

Hence, using the notation that we introduced in Section 7 above

$$\overset{v}{\nabla}{}^{\mathcal{E}}(u^k\pi^*e_k) = \Pi_{\mathcal{V}}[(\pi^*\nabla^{\mathcal{E}})(u^k\pi^*e_k)]^\sharp = \pi_{\mathcal{V}}(du^k)^\sharp \otimes e_k + u^k\Gamma^j_{ik}\left(\pi_{\mathcal{V}}\left(\pi^*dx^i\right)^\sharp \otimes \pi^*e_j\right).$$

By definition we have that $\pi_{\mathcal{V}}(du^k)^\sharp = \overset{v}{\nabla}u^k$. Next let's show that each $\pi_{\mathcal{V}}\left(\pi^*dx^i\right)^\sharp = 0$. Take any $W \in \mathcal{V}$. Using that $TSM = \mathbb{R}X \oplus \mathcal{H} \oplus \mathcal{V}$ is an orthogonal decomposition, we have that

$$\langle \pi_{\mathcal{V}}\left(\pi^*dx^i\right)^\sharp, W\rangle = \langle\left(\pi^*dx^i\right)^\sharp, W\rangle = \pi^*dx^i(W).$$

It's not hard to see that W being vertical implies that it each $dx^i(W) = 0$. So indeed $\pi_{\mathcal{V}}\left(\pi^*dx^i\right)^\sharp = 0$. From here we finally get (10.1).

Next, I believe that for any $Z \in \mathcal{Z}$ and any e_i

$$\overset{v}{\mathrm{div}}{}^{\mathcal{E}}(Z \otimes e_i) = \overset{v}{\mathrm{div}}(Z)e_i. \tag{10.2}$$

The reason I believe this to be true is that the computation for the vertical divergence in Appendix A of the reference "[PSU14c]" mentioned in the paper should adapt easily to prove the above formula. If that's the case, then the highlighted equation simply follows by applying the $-\overset{v}{\mathrm{div}}{}^{\mathcal{E}}$ to both sides of (10.1) above.

11 Page 105 (PDF page 23) Calculation in Proof of Lemma 4.2

Using the inequality $|(F^{\mathcal{E}}u, \overset{v}{\nabla}{}^{\mathcal{E}}u)| \leq \frac{1}{2}(\frac{1}{\kappa}\|F^{\mathcal{E}}\|^2_{L^\infty}\|u\|^2 + \kappa\|\overset{v}{\nabla}{}^{\mathcal{E}}u\|^2)$, we get

In this section I'd like to show why this highlighted inequality follows. Here $(\cdot,\cdot)$ and $\|\cdot\|$ denote the inner product and norm respectively in the space $L^2(SM; N \otimes \mathcal{E})$. First we observe that

$$0 \leq \left\|\frac{1}{\sqrt{\kappa}}F^{\mathcal{E}}u \pm \sqrt{\kappa}\overset{v}{\nabla}{}^{\mathcal{E}}u\right\|^2 = \left\|\frac{1}{\sqrt{\kappa}}F^{\mathcal{E}}u\right\|^2 \pm 2\left(\frac{1}{\sqrt{\kappa}}F^{\mathcal{E}}u, \sqrt{\kappa}\overset{v}{\nabla}{}^{\mathcal{E}}u\right) + \left\|\sqrt{\kappa}\overset{v}{\nabla}{}^{\mathcal{E}}u\right\|^2$$

and so

$$\left|\left(F^{\mathcal{E}}u, \overset{\mathrm{v}}{\nabla}^{\mathcal{E}}u\right)\right| \leq \frac{1}{2}\left(\frac{1}{\kappa}\left\|F^{\mathcal{E}}u\right\|^2 + \kappa\left\|\overset{\mathrm{v}}{\nabla}^{\mathcal{E}}u\right\|^2\right).$$

So we need to show that

$$\left\|F^{\mathcal{E}}u\right\|^2 \leq \left\|F^{\mathcal{E}}\right\|_{L^\infty}^2 \|u\|_{L^2(SM;\mathcal{E})}^2.$$

It might be useful to review Section 9 above for the following. We will prove the above inequality by showing that at any point $(x, v) \in SM$

$$(11.1) \qquad \left|F^{\mathcal{E}}u\right|_{N\otimes\mathcal{E}}^2 \leq \left|F^{\mathcal{E}}\right|_{N\otimes\mathcal{E}^*\otimes\mathcal{E}}^2 |u|_{\mathcal{E}}^2.$$

Fix $(x, v) \in SM$ and let (b_i) be an orthonormal basis of N_x and (r_j) be an orthonormal basis of $\mathcal{E}_x$. Let (β^i) and (ρ^j) be dual bases for (b_i) and (r_j) respectively. In such frames we write

$$F^{\mathcal{E}} = (F^{\mathcal{E}})^i{}_r{}^j\, b_i \otimes \rho^r \otimes r_j \qquad \text{and} \qquad u = u^r r_j,$$

and so

$$F^{\mathcal{E}}u = (F^{\mathcal{E}})^i{}_r{}^j u^r b_i \otimes r_j.$$

Since our frames are all orthonormal, we can write the norm of this rather neatly:

$$\left|F^{\mathcal{E}}u\right|_{N\otimes\mathcal{E}}^2 = \sum_{i=1}^{d-1}\sum_{j=1}^{n}\left[(F^{\mathcal{E}})^i{}_r{}^j u^r\right]^2.$$

Using the triangle inequality on the summand here gives us that this is bounded by

$$\sum_{i=1}^{d-1}\sum_{j=1}^{n}\left(\sum_{r=1}^{n}\left[(F^{\mathcal{E}})^i{}_r{}^j\right]^2\right)\left(\sum_{r=1}^{n}[u^r]^2\right) = \left|F^{\mathcal{E}}\right|_{N\otimes\mathcal{E}^*\otimes\mathcal{E}}^2 |u|_{\mathcal{E}}^2,$$

proving (11.1) above.

12 Page 105 (PDF Page 22) 2nd Calculation in Proof of Lemma 4.2

Finally, we have $\frac{\|F^{\mathcal{E}}\|_{L^\infty}^2}{2\kappa}\|u\|^2 \leq \frac{\|F^{\mathcal{E}}\|_{L^\infty}^2}{2\kappa\lambda_m}\|\overset{\mathrm{v}}{\nabla}{}^{\mathcal{E}}u\|^2$, and thus if m is so large that

$$\lambda_m \geq \frac{2\|F^{\mathcal{E}}\|_{L^\infty}^2}{\kappa^2},$$

I don't quite know how the highlighted equation follows, but let me discuss a possible explanation. More explicitly, let's show that

$$\|u\|^2_{L^2(SM;\mathcal{E})} \leq \frac{1}{\lambda_m}\left\|\overset{\mathrm{v}}{\nabla}^{\mathcal{E}} u\right\|_{L^2(SM;N\otimes\mathcal{E})}$$

(recall that by assumption u_j for $j < m$ are zero). We have that

$$(12.1)\qquad \|u\|^2_{L^2(SM;\mathcal{E})} = \sum_{j=m}^{\infty}\langle u_j, u_j\rangle_{L^2(SM;\mathcal{E})} = \sum_{j=m}^{\infty}\frac{1}{\lambda_j}\langle \Delta^{\mathcal{E}} u_j, u_j\rangle_{L^2(SM;\mathcal{E})}$$

$$= \sum_{j=m}^{\infty}\frac{1}{\lambda_j}\langle \overset{\mathrm{v}}{\nabla}^{\mathcal{E}} u_j, \overset{\mathrm{v}}{\nabla}^{\mathcal{E}} u_j\rangle_{L^2(SM;N\otimes\mathcal{E})} \leq \frac{1}{\lambda_m}\sum_{j=m}^{\infty}\left\|\overset{\mathrm{v}}{\nabla}^{\mathcal{E}} u_j\right\|^2_{L^2(SM;N\otimes\mathcal{E})}.$$

Now, all of the $\overset{\mathrm{v}}{\nabla}^{\mathcal{E}} u_j$'s are orthonormal with respect to $L^2(SM; N\otimes\mathcal{E})$ since

$$\langle \overset{\mathrm{v}}{\nabla}^{\mathcal{E}} u_j, \overset{\mathrm{v}}{\nabla}^{\mathcal{E}} u_k\rangle = \langle \Delta^{\mathcal{E}} u_j, u_k\rangle = \lambda_j\langle u_j, u_k\rangle = 0.$$

Hence the last item in (12.1) above is equal to

$$\frac{1}{\lambda_m}\left\|\sum_{j=m}^{\infty}\overset{\mathrm{v}}{\nabla}^{\mathcal{E}} u_j\right\|_{L^2(SM;N\otimes\mathcal{E})}.$$

Now, the claim will follow if we could show that this is equal to

$$\frac{1}{\lambda_m}\left\|\overset{\mathrm{v}}{\nabla}^{\mathcal{E}} u\right\|_{L^2(SM;N\otimes\mathcal{E})},$$

but I'm not quite sure how to rigorously prove this.

13 Page 107 (PDF Page 25) Interchanging $\mathbb{X}$ and Sum in Fourier Series

> *Second proof of Theorem 4.1.* Let $\mathbb{X}u = f$ where f has degree l. Looking at Fourier coefficients we have $(\mathbb{X}u)_k = 0$ for $k \geq l+1$, meaning that
>
> $$\mathbb{X}_+ u_k = -\mathbb{X}_- u_{k+2}, \quad k \geq l.$$

In this step, the authors are implicitly interchanging $\mathbb{X}$ and the summation symbol in the Fourier series for u:

$$\mathbb{X}\sum_{m=0}^{\infty} u_m = \sum_{m=0}^{\infty}\mathbb{X}u_m.$$

In this note I'd like to justify this step. For simplicity, we will instead prove the following fact:

$$(13.1) \qquad X\sum_{m=0}^{\infty} u_m = \sum_{m=0}^{\infty} Xu_m, \qquad \forall u \in C^{\infty}(SM).$$

By looking in an orthonormal frame of $\mathcal{E}$, it's shouldn't be hard to see how the former claim will follow from (13.1). To prove (13.1), take any $u \in C^{\infty}(SM)$. Take any point x_0. We will prove that the above equation holds at x_0. Consider normal coordinates (x^i) centered at x_0 which naturally generate the coordinates $v^j\,\partial/\partial x^j \mapsto (x^i, v^j)$ of TM. Let (g_{ij}) denote the metric tensor in these coordinates. Above the point x_0 we have that $X = v^i\,\partial/\partial x^i$. So the claim will follows if we can justify interchanging $\partial/\partial x^i$ with the summation sign. Unfortunately, doing this in our coordinates of TM is a little inconvenient, so we construct another set of coordinates.

Let (b_i) be the smooth orthonormal frame over the domain of (x^i) obtained by applying the Gram-Schmidt orthogonalization process to the frame $(\partial/\partial x^i)$. This frame gives us another set of coordinates of TM given by $w^j b_j \mapsto (x^i, w^j)$. Let (α_μ^ν) be the coefficients in the relation $b_\mu = \alpha_\mu^\nu\,\partial/\partial x^\nu$. Thinking about how the Gram-Schmidt orthogonalization process works, it's not hard to see that each $\partial g_{ij}/\partial x^r$ being equal to zero at $x = x_0$ implies that all of the partials $\partial\alpha_\mu^\nu/\partial x^r$ are zero at $x = x_0$ as well (hint: use induction). Furthermore, if we let (β_μ^ν) be the coefficients in the inverse relation $\partial/\partial x^\mu = \beta_\mu^\nu b_\nu$, it's not hard to see that the β_μ^ν's share the same property of the α_μ^ν's mentioned in the previous sentence. From this observation we see that above x_0, $X = w^i\,\partial/\partial x^i$. So we simply need justify interchanging the partial $\partial/\partial x^i$ taken with respect to (x^i, w^j) with the summation sign.[1]

For every integer $m \geq 0$, let $\{Y_{m,r} : r = 1, \dots, l_m\}$ be a set of real harmonic polynomials homogeneous of degree m over $\mathbb{R}^n$ with respect to the (flat) Euclidean Laplacian such that they span the spherical harmonics of order m over S^{d-1}. For such m and r, consider the smooth functions $\{\mathcal{Y}_{m,r}\}$ defined over TM near x_0 given by

$$\mathcal{Y}_{m,r}\big(x, w^i b_i\big) = Y_{m,r}(w^1, \dots, w^n).$$

Now, we have that each

$$u_m(x,w) = \sum_{r=1}^{l_m}\left[\int_{S_{x_0}M} u(x,w')\mathcal{Y}_{m,r}(x,w')\,dw'_{S_{x_0}M}\right]\mathcal{Y}_{m,r}(x,w).$$

From the previous equation we see that $\mathcal{Y}_{m,r}$ has no dependence on x and so

$$\frac{\partial u_m}{\partial x^i}(x,w) = \sum_{r=1}^{l_m}\left[\int_{S_{x_0}M} \frac{\partial u}{\partial x^i}(x,w')\mathcal{Y}_{m,r}(x,w')\,dw'_{S_{x_0}M}\right]\mathcal{Y}_{m,r}(x,w).$$

---

[1] This is a different task from before since, except at $x = x_0$, the partial $\partial u_m/\partial x^i$ is not necessarily the same thing with respect to the coordinates (x^i, v^j) and (x^i, w^j).

Now, in the integral we can write $\mathcal{Y}_{m,r}$ as $\Delta\mathcal{Y}_{m,r}/(m(m+d-2))$ and then transfer the "Δ" to the $\partial u/\partial x^i$ via integration by parts. We can do this several times, say k times where $k \geq 1$ is an integer. After doing so, we get the following estimate where ω_{d-1} denotes the surface area of S^{d-1}:

$$\left|\frac{\partial u_m}{\partial x^i}(x,w)\right| \leq \sum_{r=1}^{l_m} \frac{\sup_{w\in S_xM}\left|\Delta^k \frac{\partial u}{\partial x^i}(x,w)\right| \left(\sup|\mathcal{Y}_{m,r}|\right)^2 \omega_n}{\left(m(m+d-2)\right)^k}$$

$$= \frac{\sup_{w\in S_xM}\left|\Delta^k \frac{\partial u}{\partial x^i}(x,w)\right|}{\left(m(m+d-2)\right)^k}\omega_n \sum_{r=1}^{l_m}\left(\sup|\mathcal{Y}_{m,r}|\right)^2.$$

By well-known results, the last "$\sum$ ..." sum is bounded by a polynomial in m (for instance, this follows immediately from Corollary 2.55 and Theorem 2.57 parts (a) and (f) in [1] – simply set $y = x$ into $Z_k^x(y)$ in part (a) there). Furthermore, the supremum of $\Delta^k\, \partial u/\partial x^i$ can be bounded near x_0. So, by choosing k large enough, in a neighborhood of x_0 we can bound $\partial u_m/\partial x^i\,(x,w)$ by a polynomial in m independent of x and w that is finitely summable in m as $m \to \infty$. This justifies interchanging $\partial/\partial x^i$ and the summation symbol in (13.1), and hence we've proven what we wanted.

14 Page 109 (PDF Page 27) Remainder Estimate in Proof of Theorem 4.6

> Let $m = N+1+2k$, where k is a non-negative integer and N is an integer with $N \geq N_0$. Note that from the definition of r_m and our choice of N we have
>
> $$r_m + r_{m-2} + \cdots + r_{N+1} \geq -B\|u_N\|^2.$$

I'd like to give a quick note on how this inequality follows. We have that

$$r_m + r_{m-2} + \cdots + r_{N+1} =$$

$$-B\|u_{m-1}\|^2 + c_{m+1}\|u_{m+1}\|^2 + (c_m - C)\|u_m\|^2$$

$$-B\|u_{m-3}\|^2 + c_{m-1}\|u_{m-1}\|^2 + (c_{m-2} - C)\|u_{m-2}\|^2$$

$$- \cdots$$

$$-B\|u_N\|^2 + c_{N+2}\|u_{N+2}\|^2 + (c_{N+1} - C)\|u_{N+1}\|^2.$$

Now, since the c_k's are assumed to bigger than C, to bound this quantity from below we can simply remove the terms $(c_k - C)\|u_k\|^2$. Since the c_k are assumed to be bigger than B, then we can similarly remove the terms $-B\|u_k\|^2 + c_k\|u_k\|^2$. The only thing that will be left is $c_{m+1}\|u_{m+1}\|^2 - B\|u_N\|^2$. For the same reason, we can remove $c_{m+1}\|u_{m+1}\|^2$ to get the desired inequality.

15 Page 119 (PDF Page 37) Final Conclusion in Proof of Theorem 1.2

> and thus $I_{\hat{\nabla},\hat{\Phi}}(\tilde{A} + \tilde{\Phi} - \Phi) = 0$ in the bundle $\mathcal{F}$, which implies by Theorem 1.1 that there exists $Z \in C^\infty(M;\mathcal{F})$ vanishing at ∂M such that
>
> $$\mathbb{X}Z - Z\tilde{A} + \Phi Z - Z\tilde{\Phi} = \tilde{A} + \tilde{\Phi} - \Phi$$
>
> and $W = Z$ and $Q = Z + \mathrm{Id}$ gives the desired gauge equivalence. $\square$

In this note I'd like to explain how the Q that they obtain gives the desired gauge. Take the second to last equation before this in the paper:

$$\mathbb{X}Q + \Phi Q - Q\tilde{A} - Q\tilde{\Phi} = 0, \quad U|_{\partial(SM)} = \mathrm{id},$$

and multiply through by an arbitrary smooth $V \in \mathcal{F}$ on the right. Rearranging and adding $Q\mathbb{X}V$ gives

$$\mathbb{X}(Q)V + Q\mathbb{X}V + \Phi QV = Q\mathbb{X}V + Q\tilde{A}V + Q\tilde{\Phi}V,$$

$$Q^{-1}\mathbb{X}(QV) + Q^{-1}\Phi QV = \mathbb{X}V + \tilde{A}V + \tilde{\Phi}V,$$

$$Q^{-1}(\mathbb{X} + \Phi)(QV) = \mathbb{X}V + \tilde{A}V + \tilde{\Phi}V.$$

Fixing $x \in M^{\mathrm{int}}$, we see that both sides of the equation are functions of the matrix V and $v \in S_xM$. We can extend both sides to $v \in T_xM$ by setting the terms that contain v to be homogeneous of order one and terms that don't contain v to be homogeneous of order zero. From there we see that the two types of terms must be equal to their respective counterpart on the other side of the equation, giving us that

$$Q^{-1}\mathbb{X}(QV) = \mathbb{X}V + \tilde{A}V,$$

$$Q^{-1}\Phi QV = \tilde{\Phi}V,$$

Clearly the second condition tells us that $Q^{-1}\Phi Q = \tilde{\Phi}$. We claim that the first equation implies that $Q^{-1}\nabla Q = \tilde{\nabla}$ as an equation of connections on M. To see why, apply both sides of that equation to a smooth field[2] $f' \in \pi^*\mathcal{E}$ to get

$$Q^{-1}[(\pi^*\nabla_X)(QVf) - QV(\pi^*\nabla_X)f] = \left(\pi^*\tilde{\nabla}_X\right)(Vf) - V(\pi^*\nabla_X)f,$$

Which after relabeling $Vf \mapsto f$ gives

$$Q^{-1}(\pi^*\nabla_X)(Qf) = \left(\pi^*\tilde{\nabla}_X\right)f.$$

Now consider f of the form $f = \pi^*f'$ where $f' : M \to \mathcal{E}$ is smooth. At any point $(x,v) \in SM$ the left-hand side of this equation becomes

[2] I will write in the (typically omitted) π^* in the section because it will be important here to distinguish what's happening on SM and down in M.

$$Q^{-1}(\pi^*\nabla)_X\big(\pi^*(Qf')\big) = Q^{-1}\pi^*\big(\nabla_v(Qf')\big) = \pi^*\big(Q^{-1}\nabla_v(Qf')\big),$$

while the right-hand is similarly equal to $\pi^*\widetilde{\nabla}_v(f')$. Hence we indeed get that $Q^{-1}\nabla Q = \widetilde{\nabla}$ as an equation of connections on M.

16 References

The following are references used in addition to the paper being analyzed in this document:

1. Folland, G. B. (1995). *Introduction to Partial Differential Equations* (2nd ed.). Princeton: Princeton University Press.