Dear Dr. Paul J. Nahin,

Hi Dr. Nahin (I didn't know how to address you so I went with "Dr. Nahin"), I am a student at the University of Washington and I have read both of your books "An Imaginary Tale: The Story of $\sqrt{-1}$" and "Dr. Euler's Fabulous Formula." I would like to tell you that I loved both of your books very much. In this letter I would like to explain to you what I enjoyed about these books.

Before I dive in, let me explain a little bit about who I am. Before eighth grade I was just a curious mind. In eighth grade however my mother signed me up to an after-school math circle where I got to learn mathematics for about 2 years. I came to really love mathematic and ever since I have been doing a lot of mathematics on my own. I have for example discovered a lot of mathematical formulas and theorems for myself. I have come to know mathematics as another dimension of logic that I can escape to at any time I want. I do mathematics almost every day as it has come to define a very large portion of who I am.

Oh don't worry… I have other things that define me too. I have hobbies and people I love. I experience the outdoors. For example, I have been on top of Mt. Rainier, Mt. Baker and other volcanic peaks in our area. I do a lot of hiking in the cascades. I have studied a lot of physics and computer science as well. I read books and write. So I'm not all math. But back to math though:

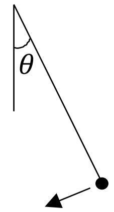
* Note: If in this letter you see the symbol "*" in the equations it means "multiply," not "convolution." Convolutions do not appear anywhere in this letter. I tried to change all of the "*" symbols into "·", but I may have missed some.

<u>Part I</u>

Finding the Book

I read the first book "An Imaginary Tale: The Story of $\sqrt{-1}$" when I was in high school. One day when our class was visiting our school library (for some research project probably) I decided to amuse myself by looking at what physics and math books they had there. I first picked up a calculus physics book (which was weird because our school seemed to frown upon the use of calculus in science) in which I turned to the page which had the following statement (reworded):

"Some smart person proved that no elementary function satisfies the following differential equation for the pendulum ($g \approx 9.8\, m/s^2$):"



$$L \cdot \frac{d^2\theta}{dt^2} = g \cdot \sin(\theta)$$

All I said was "Wow! I don't know how to prove that" and put the book back on its shelf. Then I picked up another book titled "An Imaginary Tale: The Story of $\sqrt{-1}$." I looked inside and saw lots of these kinds of symbols: $\prod$ and so I immediately told myself that I've got to read this book! I checked it out of the library and decided to take it home to read. I thought if I won't understand something I can always ask my dad for help (my dad knows a lot of math).

For what follows, page numbers in "An Imaginary Tale: The Story of $\sqrt{-1}$" refer to the edition where your preface to the paperback edition was written in 2006.

The Cubic Equation

I started to read your book and I quickly got hooked. At first I was really amazed at the fact that in just the first 3 pages you derive the cubic formula. I always considered it really difficult. In fact, when I was much younger I printed it out and glued a copy of it into my mathematics journal. The version I printed out looked very big and very complicated. But you gave a very easy and beautiful derivation of it that del Ferro thought of a long time ago. I had tried to derive the cubic formula myself for a long time before I read your book, and I was not successful. In your book you started off with the cubic equation:

$$x^3 + px = q$$

Immediately somebody might have said "But wait! That isn't the general cubic equation!" But I knew better. You can always get from the general cubic equation:

$$ax^3 + bx^2 + cx + d = 0$$

To the previous cubic equation by scaling and shifting (I'll elaborate a little bit later when I'll present to you the full cubic equation). Returning to solve the first cubic equation above you proceed by doing a trick (on page 9) of saying "let $x = u + v$" and then you set (I skip steps):

$$3uv + p = 0$$

$$u^3 + v^3 = q$$

And pretty quickly you arrive at the solution:

$$x = \sqrt[3]{\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} - \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

Obviously when you get one root of a cubic equation, you can get the other two by dividing the original cubic equation by x minus the first root and then use the quadratic formula in order to obtain the other two roots.

Later the cubic equation got involved in a rather amusing story. When I became a student at the University of Washington, I immediately signed up for Honors Calculus 1 (It was a really good class). One time our teacher gave us a homework problem which involved solving a cubic equation. He gave us the hint of just completing the cube (he gave us a really nice cubic equation) in order to solve it. But I of course did not do that. I decided to show off my skills by deriving the full cubic formula in the homework and plugging in the coefficients of the cubic equation into it. The formula I presented in the homework was:

"One solution of the general cubic equation $ax^3 + bx^2 + cx + d = 0$ is:

$$x = \sqrt[3]{-\frac{\frac{d}{a} + \frac{2\left(\frac{b}{a}\right)^3}{27} - \frac{bc}{3a^2}}{2} + \sqrt{\frac{\left(\frac{d}{a} + \frac{2\left(\frac{b}{a}\right)^3}{27} - \frac{bc}{3a^2}\right)^2}{4} + \frac{\left(\frac{c}{a} - \frac{\left(\frac{b}{a}\right)^2}{3}\right)^3}{27}}} + \sqrt[3]{-\frac{\frac{d}{a} + \frac{2\left(\frac{b}{a}\right)^3}{27} - \frac{bc}{3a^2}}{2} - \sqrt{\frac{\left(\frac{d}{a} + \frac{2\left(\frac{b}{a}\right)^3}{27} - \frac{bc}{3a^2}\right)^2}{4} + \frac{\left(\frac{c}{a} - \frac{\left(\frac{b}{a}\right)^2}{3}\right)^3}{27}}} - \frac{b}{3a}$$

You can actually recognize certain elements of the formula you gave in your book on page 10 in this formula. The $\frac{b}{3a}$ term is the shift needed to delete the quadratic term in the general cubic equation. Meaning that after you divide through $ax^3 + bx^2 + cx + d = 0$ by a and then plug in $x = y - \frac{b}{3a}$ you will get a cubic equation of the form $y^3 + py = q$, and then you can proceed as you do in your book.

When he returned our graded homework the next Friday, I came up to him and asked him what did he think of my derivation of the cubic formula. He looked at me with wide eyes (obviously lack of sleep) and said… that I need to take harder classes. He said that he was not able to follow all of the steps (he probably graded our homework late at night) and just looked at whether my answer was correct.

Later I told a graduate student how to solve the cubic equation, and the graduate student didn't know del Ferro's method of solving the cubic equation. He like everybody else kept showing me in return a much more complicated method of solving the cubic equation. I'm still surprised that no one seems to know the approach that you show in your book in solving the cubic equation.

Formulas I can't get over

You know, there are only a few formulas in mathematics that I just can't ever get over. Those are formulas that, when I learn them, come at me so unexpectedly and inspire much awe. Three such formulas that I have yet not been able to get over are:

$$\frac{\sin(\theta)}{\theta} = \prod_{k=1}^{\infty} \cos\left(\frac{1}{2^k}\theta\right)$$

(page 64) and its corollary

$$\prod_{k=1}^{\infty} \cos\left(\frac{1}{2^k}\theta\right) = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2+\sqrt{2}}}{2} \cdot \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2} \cdot \ldots = \frac{2}{\pi}$$

and the result (page 72):

$$\prod_{k=1}^{n-1} \sin\left(\frac{k\pi}{2n}\right) = \prod_{k=1}^{n-1} \cos\left(\frac{k\pi}{2n}\right) = \frac{\sqrt{n}}{2^{n-1}}$$

The last result is really amazing to me because I know how to sum cosines (in more ways than one). I can even take the limit of an infinitesimal times a sum of cosines on a finite interval (which is called the "integral"). But to be able to take such a product! That was completely unexpected.

Complex Geometry (Pun Intended)

The history you provide in your book is fascinating. It was especially interesting to learn about how people for a long time have tried to grapple with the geometric meaning of the square root of negative one (which I will henceforth denote by i). It was really funny to read about how (I unfortunately can't find the page) one person (who studied ballistics) was disapproving of Wessel's idea of representing complex numbers as vectors in a plane because he didn't like mixing geometry and algebra. To me that sounds silly because I always get excited when fields of mathematics merge together. I also found it really cool that you can use complex numbers to prove geometric facts such as the medians of any triangle meet at one point (page 89) and you can even say how far that point is from each of the vertices of the triangle. Complex numbers became a powerful tool of mine in solving all kinds of geometric problems.

Where did that come from?

In Chapter 4 you discuss the Leonardo Reoccurrence, which at the time blew my mind! It still does. My favorite phrase in that section is "I will start by guessing $u_n = k \cdot z^n$." There you defined $\{u_n\}$ is a sequence defined recursively by the formula $u_{n+2} = p \cdot u_{n+1} + q \cdot u_n$ with initial conditions u_0 and u_1. Of course the instant thought that comes to the mind of the reader is "how did somebody come up with that guess!" After reading a little bit further the reader is then amazed at how you, with this guess in hand, are able to derive a formula for u_n. That was exactly what was going through my mind. Though, this mystery of where the idea came from to assume $u_n = k \cdot z^n$ didn't last forever with me.

One time my dad tried to explain to me how matrices can be used to arrive at the guess just mentioned about the Leonardo Reoccurrence. Although I did not understand what he was talking about, his words stuck with me. When I became a freshman at the University of Washington, at one point (after I gained some experience with matrices) I realized that the Leonardo Reoccurrence can be written down in the form of matrix-vector multiplications (in fact the solution to systems of linear differential equations borrow this idea as well!). We could in fact write down the Leonard Reoccurrence in the following way:

If $u_{n+2} = p \cdot u_{n+1} + q \cdot u_n$

Then let us write down the following sequence of vectors:

$$\left\{\begin{bmatrix} u_{2k} \\ u_{2k+1} \end{bmatrix}\right\}_{k=0}^{\infty} = \left\{\begin{bmatrix} u_0 \\ u_1 \end{bmatrix}, \begin{bmatrix} u_2 \\ u_3 \end{bmatrix}, \begin{bmatrix} u_4 \\ u_5 \end{bmatrix}, \dots\right\}$$

Then we can notice that the reoccurrence can be rewritten in the following manner in matrix-vector form (you can check that this is true by just multiplying it out):

$$\begin{bmatrix} q & p \\ pq & p^2+q \end{bmatrix}\begin{bmatrix} u_n \\ u_{n+1} \end{bmatrix} = \begin{bmatrix} u_{n+2} \\ u_{n+3} \end{bmatrix}$$

I will admit that this is a guess too, but … anyways, after one writes down the previous recursive formula, it is almost an immediate result that:

$$for\ n\ even, \qquad \begin{bmatrix} u_n \\ u_{n+1} \end{bmatrix} = \begin{bmatrix} q & p \\ pq & p^2+q \end{bmatrix}^{\frac{n}{2}}\begin{bmatrix} u_0 \\ u_1 \end{bmatrix}$$

Thus can we see were the guess you made in the book $u_n = k \cdot z^n$ comes from? It comes from the above equation. Of course you still need to be able to take matrices to high powers. And I knew just how to do that. In fact it was at this time that I was reading the section 1.2 in your book “Dr. Euler’s Fabulous Formula” where you talked about how to take matrices to high powers. I haven’t even started studying linear algebra at the University of Washington! In section 1.2 in “Dr. Euler’s Fabulous Formula,” you demonstrate the fact that for a 2 by 2 matrix A (I is the identity matrix):

$$A^n = (k_1\lambda_1^n + k_2\lambda_2^n)A + (k_3\lambda_1^n + k_4\lambda_2^n)I$$

For some constants, possibly complex, $k_1, k_2, k_3, k_4, \lambda_1, \lambda_4$ (you show how to calculate these constants in “Dr. Euler’s Fabulous Formula”). Now if we say that $A = \begin{bmatrix} q & p \\ pq & p^2+q \end{bmatrix}$, then from the two formulas above we get that

$$\begin{bmatrix} u_n \\ u_{n+1} \end{bmatrix} = \left(\left(k_1\lambda_1^{n/2} + k_2\lambda_2^{n/2}\right)\begin{bmatrix} q & p \\ pq & p^2+q \end{bmatrix} + \left(k_3\lambda_1^{n/2} + k_4\lambda_2^{n/2}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} u_0 \\ u_1 \end{bmatrix}$$

For some constants $k_1, k_2, k_3, k_4, \lambda_1, \lambda_4$. Now if we multiply out the right hand side in such a way as to only get the first component of the vector on the left hand side of the equation we get:

$$u_n = u_0\left(\left(k_1\lambda_1^{n/2} + k_2\lambda_2^{n/2}\right)q + \left(k_3\lambda_1^{n/2} + k_4\lambda_2^{n/2}\right)\right) + u_1\left(\left(k_1\lambda_1^{n/2} + k_2\lambda_2^{n/2}\right)p\right)$$

And collecting terms will give us that u_n can be expressed as:

$$u_n = k_5\sqrt{\lambda_1}^n + k_6\sqrt{\lambda_2}^n$$

For some constants k_5, k_6. This is exactly the form that you present your solution to the Leonardo Reoccurrence in, and this is an origin of where the idea that u_n possibly grows as an exponential function comes from. With this guess in hand, one can now solve the Leonardo Reoccurrence in exactly the same manner as you did in your book.

I want to note that my dad noticed that my reoccurrence matrix could be much simpler if I instead considered the sequence:

$$\left\{\begin{bmatrix} u_k \\ u_{k+1} \end{bmatrix}\right\}_{k=0}^{\infty} = \left\{\begin{bmatrix} u_0 \\ u_1 \end{bmatrix}, \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \begin{bmatrix} u_2 \\ u_3 \end{bmatrix}, \dots\right\}$$

In this case the matrix – vector form of the Leonardo Reoccurrence becomes:

$$\begin{bmatrix} 0 & 1 \\ p & q \end{bmatrix}\begin{bmatrix} u_n \\ u_{n+1} \end{bmatrix} = \begin{bmatrix} u_{n+1} \\ u_{n+2} \end{bmatrix}$$

And so our matrix is instead:

$$\begin{bmatrix} 0 & 1 \\ p & q \end{bmatrix}$$

This matrix is much easier to handle (such as finding its eigenvalues and eigenvectors) than the matrix that I presented.

Something for Later

I got to the point in your book when you started to talk about how imaginary numbers play a role in the theory of relativity. I did not understand the section very much at that time, and so I skipped it and said to myself that I would return to it after I've read about the theory of relativity in the Feynman Lectures (it was approximately at that time that I started to read the Feynman Lectures on Physics). The same went for the uses of complex numbers in electrical circuits.

Celestial Mechanics

The section 5.3 is probably my favorite section of the whole book. Since I was little, I've always known about the 3 laws of Kepler, but I never knew how to prove them using Newton's Gravitational law: $F_G = G\frac{mM}{r^2}$. I remember how when I was reading the beginning of this section I was telling myself, "No, you're not going to derive the 3 laws of Kepler. That just couldn't be. I heard that they're really hard to prove." And just like with your derivation of the cubic formula, you surprised and amazed me by deriving what I always thought was really difficult to prove. You showed once again how complex numbers can give beautiful derivations of laws that would otherwise be hard to prove.

About a year later after I read this section I wondered what would happen if I would have used another gravitational law. For example what would have happened if I used the gravitational law:

$$F_G = G\frac{mM}{r^3}$$

How would the orbit of a planet be changed if there was a cube in the denominator? Well what I did was I went back and I carried out the calculation just the way you did in your book. At one point I got to, just like you did on page 117, the formula

$$\frac{dA}{dt} = \frac{1}{2}c_1$$

Where A represents the function of how much area the planet sweeps out versus time. This was your proof of Kepler's second gravitational law. I realized that this equation was in fact invariant of the gravitational law that I picked in the first place. So Kepler's second law was true for any gravitational force. When I told this to my dad, he almost immediately said: "Well yeah, that is just the conservation of angular momentum." And then I right away understood that Kepler's second law was true by the mere fact that the gravitational force is always pointing towards the sun (or a planet that it is orbiting) and so there is no torque in the system of its rotation. This means that the angular momentum of the planet around the sun is always constant. Now angular momentum (usually denoted by L) is given by the formula:

$$L = I\omega$$

Where I is the moment of inertia of the system and ω is the angular speed of the planet. Since in this system $I = mr^2$ and by definition: $\omega = \theta'(t)$, we have that:

$$L = mr^2 \frac{d\theta}{dt}$$

And since we said that L is constant (say $L = c_2$) and from freshman calculus we know that $A'(t) = \frac{1}{2}r^2\frac{d\theta}{dt}$, we have that:

$$\frac{dA}{dt} = \frac{c_2}{m}$$

So $A'(t)$ is constant. This means that Kepler's second law holds under any gravitational force as long as the force is always pointed along the line that connects the sun and the planet. I was not able to arrive at a good answer for how the orbit would look like if the gravitational force was not proportional to the inverse of the square of the distance between the two planets.

Wow!

Wow! Sorry, I couldn't help myself. I was amazed with Euler's derivation of $\zeta(2)$ when I read your book. In fact, if the equation $e^{\pi i} = -1$ was Feynman's "remarkable formula," my "remarkable formula" would be non-other than:

$$\zeta(2) = \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$$

To me I have never seen such an outstanding, remarkable, and beautiful formula in all of my life! In fact this formula has taken up so much of my focus that I have thought of 6 or 7 additional separate proofs of this formula. On pages 148-149, you show how Euler originally derived this fact. I then realized that what you said was true: that the very same method can be used to derive the value of $\zeta(4)$ and the rest of the values of the zeta function at even integers. Let me show how I was able to extend Euler's method of calculating $\zeta(2)$ to compute $\zeta(4)$:

Just like you do in your book, I started off with the Taylor Series for the sine function:

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

Then by substituting $\sqrt{x}$ into both sides of the equation and then dividing both sides by $\sqrt{x}$, one will get:

$$\frac{\sin(\sqrt{x})}{\sqrt{x}} = 1 - \frac{x}{3!} + \frac{x^2}{5!} - \frac{x^3}{7!} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^k}{(2k+1)!}$$

Now just like in your book we notice that the above function has the roots $\pi^2, (2\pi)^2, (3\pi)^2$, ... Now, using the trick that a polynomial of the form:

$$P(x) = 1 + \sum_{k=1}^{n} c_k x^k$$

(Here the c_k's are constants) can be written as:

$$P(x) = (-1)^n \prod_{k=1}^{n} (1 - \frac{x}{r_k})$$

where the r_k's are the roots of $P(x)$, we can then rewrite the above series for $\frac{\sin(\sqrt{x})}{\sqrt{x}}$ as:

$$\frac{\sin(\sqrt{x})}{\sqrt{x}} = \left(1 - \frac{x}{(\pi)^2}\right)\left(1 - \frac{x}{(2\pi)^2}\right)\left(1 - \frac{x}{(3\pi)^2}\right) \ldots = \prod_{k=1}^{\infty} \left(1 - \frac{x}{(k\pi)^2}\right)$$

Now if you start multiplying this expression out using the distributive property in such a way as to get the x^2 term in the Taylor Series of this function, you will get:

$$\frac{1}{\pi^4} \sum_{k>j} \frac{1}{k^2 j^2} = \frac{1}{5!}$$

Multiply both sides by π^4 to get:

$$\sum_{k>j} \frac{1}{k^2 j^2} = \frac{\pi^4}{5!}$$

Now that we have this preliminary result in hand we go onto the main steps. Let us take the Taylor Polynomial of $\frac{\sin(\sqrt{x})}{\sqrt{x}}$ that we derived above and square both sides in order to get:

$$\left(\frac{\sin(\sqrt{x})}{\sqrt{x}}\right)^2 = \left(1 - \frac{x}{3!} + \frac{x^2}{5!} - \frac{x^3}{7!} + \cdots\right)^2$$

Now it shouldn't be too hard to show that the right hand side can be written out as:

$$\left(1-\frac{x}{3!}+\frac{x^2}{5!}-\frac{x^3}{7!}+\cdots\right)^2 = 1-\frac{2^3}{4!}x+\frac{2^5}{6!}x^2-\frac{2^7}{8!}x^3+\cdots = \sum_{k=0}^{\infty}\frac{(-1)^k x^k 2^{2k+1}}{\big(2(k+1)\big)!}$$

What is actually important to us is the fact that the coefficient of x^2 above is $\frac{2^5}{6!}$. The fact that the coefficient of x^2 is $\frac{2^5}{6!}$ is not hard to verify as a standalone fact by just directly multiplying out $\left(1-\frac{x}{3!}+\frac{x^2}{5!}-\frac{x^3}{7!}+\cdots\right)^2$ by the distributive property. If you're really interested in knowing how to prove the first equality above, it is shown by proving the following fact (which can be proved using the binomial theorem):

$$\sum_{k=0}^{\left\lceil\frac{n+1}{2}\right\rceil-1}\left(\frac{1}{(2k+1)!\,(2(n+1-k)-1)!}\right)=\frac{2^{2n-1}}{(2n)!}$$

Here $\lceil m\rceil$ denotes: "take the next bigger than m integer if m is not an integer. If m is an integer $\lceil m\rceil = m$." $\lceil m\rceil$ is called the ceiling function. But again let me remind you that for our purposes, all we need is the fact that $\frac{2^5}{6!}$ is the coefficient of x^2.

Now, let us write:

$$\left(\frac{\sin(\sqrt{x})}{\sqrt{x}}\right)^2 = \left(\left(1-\frac{x}{(\pi)^2}\right)\left(1-\frac{x}{(2\pi)^2}\right)\left(1-\frac{x}{(3\pi)^2}\right)\ldots\right)^2$$

$$= \left(1-\frac{x}{(\pi)^2}\right)\left(1-\frac{x}{(\pi)^2}\right)\left(1-\frac{x}{(2\pi)^2}\right)\left(1-\frac{x}{(2\pi)^2}\right)\ldots = \prod_{k=1}^{\infty}\left(1-\frac{x}{(k\pi)^2}\right)^2$$

And now if we multiply the expression $\prod_{k=1}^{\infty}\left(1-\frac{x}{(k\pi)^2}\right)^2$ out so as to get the x^2 coefficient of its Taylor Polynomial you will get that the x^2 coefficient is equal to:

$$\frac{1}{\pi^4}\sum_{k=1}^{\infty}\frac{1}{k^4}+4\frac{1}{\pi^4}\sum_{k>j}\frac{1}{k^2j^2}=$$

$$\frac{1}{\pi^4}\zeta(4)+4\frac{1}{\pi^4}\sum_{k>j}\frac{1}{k^2j^2}$$

There is a 4 in front of the $\frac{1}{\pi^4}\sum_{k>j}\frac{1}{k^2j^2}$ term because if you look closely at the expression $\left(1-\frac{x}{(\pi)^2}\right)\left(1-\frac{x}{(\pi)^2}\right)\left(1-\frac{x}{(2\pi)^2}\right)\left(1-\frac{x}{(2\pi)^2}\right)$ ..., when you multiply it out using the distributive property there are 4 ways to get the term $\frac{1}{(k\pi)^2(j\pi)^2}$ where $k>j$.

Now since we already know that $\frac{2^5}{6!}$ is the coefficient of the x^2 term in the Taylor Polynomial of the function $\left(\frac{\sin(\sqrt{x})}{\sqrt{x}}\right)^2$ we can then combine the two expressions for this coefficient in order to get that:

$$\frac{1}{\pi^4}\zeta(4) + 4\frac{1}{\pi^4}\sum_{k>j}\frac{1}{k^2 j^2} = \frac{2^5}{6!}$$

Multiply both sides by π^4 to get:

$$\zeta(4) + 4\sum_{k>j}\frac{1}{k^2 j^2} = \frac{2^5\pi^4}{6!}$$

But what do we do now? Now we use the preliminary result that we derived earlier. Remember that:

$$\sum_{k>j}\frac{1}{k^2 j^2} = \frac{\pi^4}{5!}$$

Plugging this into the previous expression gives us that

$$\zeta(4) + 4\frac{\pi^4}{5} = \frac{2^5\pi^4}{6!}$$

Rearranging gives:

$$\zeta(4) = \frac{2^5\pi^4}{6!} - 4\frac{\pi^4}{5!}$$

And simplifying gives us the next sequel in "remarkable formulas:"

$$\zeta(4) = \frac{\pi^4}{90}$$

Wow again! In your book "Dr. Euler's Fabulous Formula" you show yet another method of calculating these sums (specifically the zeta function at even integers) using Fourier series. But I will get to that in this letter in due course. Sorry, I have to say it one more time: wow!

I have something to say about the formula that Euler derived:

$$\frac{1}{1^3} + \frac{1}{3^3} + \frac{1}{5^3} + \cdots = \frac{\pi^2}{4}\ln(2) + 2\int_0^{\frac{\pi}{2}} x\ln(\sin(x))\,dx$$

But I will delay talking about it until I discuss a section in your book "Dr. Euler's Fabulous Formula" where you talk about a problem that Ramanujan solved.

Outstanding Results

Although I will not comment on these in depth I was fascinated to learn about Euler's constant γ, Euler's proof that $\sum_{p\ prime} \frac{1}{p}$ diverges, and the fact that $\lim_{n\to\infty} \frac{\pi(n)}{li(n)} = 1$ where $li(n) = \int_2^n \frac{dx}{\ln(x)}$. The last one, called the Prime Number Theorem, is really interesting because no matter how much I've tried, I have not yet been able to think of a proof it. I have not even been able to prove any weaker forms of this theorem. So far it's been a very frustrating problem for me.

Complex Complexities of Complicated Mathematics

I will always remember your section on Roger Cotes and a Lost Opportunity. It was so maddening to learn that a person came so close to being known as the discoverer of the formula $e^{ix} = \cos(x) + i\sin(x)$ and never did because of the fact that he didn't have clear exposition in his paper. I've definitely learned the lesson of clear exposition! I will never make this mistake.

This section started to make me think about how someone could arrive at Euler's identity using integrals, just like Cotes did. In the case of Cotes, he had a geometric interpretation for his integrals. One day when I was sitting on the bus on my way to school, I thought of a new derivation of Euler's formula involving integrals. That day before my math class started, I derived Euler's formula on the black board (I was at the University of Washington already) in front of my peers.

Often before math class started I would derive mathematical formulas on the black board in front of my classmates (the teacher usually arrived just right when the bell rung, so I had time to start my derivations before he came in). My classmates almost never listened to me, and the reason I did this was because I liked writing with chalk. I don't get to write and draw with chalk a lot. Anyways, here is the derivation of Euler's formula that I thought of:

I wondered what would happen if I integrated:

$$\int \frac{1}{1+x^2}\,dx$$

Two different ways and then equated my two results (they're supposed to give the same answer). Before we proceed, let me set a convention in this section: let C_k, for any integer k, always just denote some constant. This sounds like a reasonable convention considering that we will be doing a lot of indefinite integrations and so we will need several constants of integration.

Now, the first way to calculate the above integral is to use a result taught to all freshman calculus students:

$$\frac{d}{dx}(\tan^{-1}(x)) = \frac{1}{1+x^2}$$

$\tan^{-1}(x)$ is the arctangent function. So obvioudiously (Word doesn't seem to think that that's a word):

$$\int \frac{1}{1+x^2}\,dx = \tan^{-1}(x) + C_1$$

Where remember according to out convention C_1 is a constant (of integration). Now let us compute the integral in another way (this was in fact the period when we were studying about how to integrate rational functions, not that I didn't know how to do that already). We can do:

$$\int \frac{1}{1+x^2}dx = \int \frac{1}{(1+ix)(1-ix)}dx = \frac{1}{2}\int \left(\frac{1}{1+ix} + \frac{1}{1-ix}\right)dx$$

We know that $\int \frac{1}{y}dy = \ln{(y)}$. So the above expression becomes:

$$\frac{1}{2}\int \left(\frac{1}{1+ix} + \frac{1}{1-ix}\right)dx = \frac{1}{2i}(\ln(1+ix) - \ln(1-ix)) + C_2 = \frac{1}{2i}\ln\left(\frac{1+ix}{1-ix}\right) + C_2$$

Where C_2 is also a constant of integration. Now since these two methods of evaluating the original integral are supposed to give the same answer, we then get that:

$$\frac{1}{2i}\ln\left(\frac{1+ix}{1-ix}\right) = \tan^{-1}(x) + C_3$$

For some constant C_3. Getting the idea of where I'm heading? Ok, let us now introduce a new variables s defined by $x = \tan{(s)}$ and let us plug this new variable into the above expression. We get that:

$$\frac{1}{2i}\ln\left(\frac{1+i\tan{(s)}}{1-i\tan{(s)}}\right) = \tan^{-1}(\tan{(s)}) + C_3$$

So:

$$\ln\left(\frac{1+i\tan(s)}{1-i\tan(s)}\right) = 2is + C_4$$

For some constant C_4. Let us first get rid of the annoying C_4 term. If you plug in $s = 0$ into both sides of the equation you will get that $C_4 = 0$. So the expression becomes:

$$\ln\left(\frac{1+i\tan(s)}{1-i\tan(s)}\right) = 2is$$

Now let us simplify the expression that is sitting inside of the logarithm. Write:

$$\frac{1+i\tan(s)}{1-i\tan(s)} = \frac{\frac{\cos(s)+i\sin(s)}{\cos(s)}}{\frac{\cos(s)-i\sin(s)}{\cos(s)}} = \frac{\cos(s)+i\sin(s)}{\cos(s)-i\sin(s)} =$$

$$\frac{[\cos(s)+i\sin(s)]*[\cos(s)+i\sin(s)]}{[\cos(s)-i\sin(s)]*[\cos(s)+i\sin(s)]} = \frac{\cos^2(s)-\sin^2(s)+2\cos(s)\sin(s)\,i}{\cos^2(s)+\sin^2(s)} =$$

$$\cos(2s) + i\sin{(2s)}$$

So with the previous formula we then get that:

$$\ln(\cos(2s) + i\sin(2s)) = 2is$$

Are we warm or what! Take the exponential function of both sides to get:

$$e^{2is} = \cos(2s) + i\sin(2s)$$

And plugging in a new variable θ defined by $s = \frac{\theta}{2}$ into the above equation will give us Euler's amazing identity:

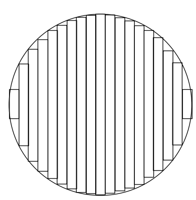
$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

I don't know what was going through my professor's head when he saw this derivation of Euler's identity, but the first thing he said was: you need to define what you mean by $\ln(1 + ix)$ and $\ln(1 - ix)$ (he just loved taking the fun out of things). I later learned that he specialized in complex analysis and so he had full authority on these things.

*After I wrote this section I looked back into the book and realized that you did a similar thing in your section "The Count Computes i^i". Here I used it in order to derive the general case, which is Euler's formula.

π

In middle school people loved π day. I was amazed at how many digits of π my classmates could memorize. I never gave it much effort since I never found it very interesting to memorize the digits of π. However one year I finally did decide to compete in this competition of who could memorize more digits of π. I believe that I placed around second place in the class, although there was no prize for that. I did however get a pencil as a participation award. But ever since 7th grade I always wondered about how people even calculated the digits of π. I mean, I could find websites with thousands upon thousands of digits of π. But how did people calculate them? My 7th grade teacher told me that his favorite method was to approximate the area of a circle by summing the following rectangles:



It was the section "Calculating π From $\sqrt{-1}$" that finaly gave me the answer. I was astounded how the simple expression $\frac{\pi i}{2} = \ln(i)$ could be used to create very quickly converging sequences that converge to $\frac{\pi}{4}$. For this all you need is to think of an expression of the form:

$$i = \frac{\prod_{k=1}^{n}(a_k + ib_k)}{\prod_{k=1}^{n}(a_k - ib_k)}$$

Where the a_k's and b_k's are real numbers. The sequence:

$$\frac{\pi}{4} = 4\left(\sum_{k=0}^{\infty}(-1)^k \frac{1}{(2k+1)5^{2k+1}}\right) - \left(\sum_{k=0}^{\infty}(-1)^k \frac{1}{(2k+1)239^{2k+1}}\right)$$

That you provide on page 174 was at that time the fastest converging sequence I ever saw to a rational multiple of π. Amazing!

*In this letter it may seem like I am skipping over certain material in your book, but I will come back to some of them later.

How to Drive People Crazy!

The name of this section is actually a hyperbole of the hyperbole. Unfortunately I was never able to evoke the "crazy" emotions I was looking for in people when I showed them the following derivation. People either ignored or just nodded their heads in silence when I showed them this mathematics (that's actually the worst reaction a presenter could see: the slow silent nod. It usually means that the person isn't following what the presenter is saying). The only person that showed some emotional reaction was my professor when he saw the following formula (which I wrote up on the blackboard… naturally):

$$\sum_{k=1}^{\infty}(-1)^k = \frac{1}{2}$$

Before I continue any further into my story, let me explain why I'm writing this section in the first place. In your book on page 185 you write that the crown jewel of mathematics is the formula (I will not disagree):

$$\zeta(s) = \zeta(1-s)\Gamma(1-s)2^s\pi^{s-1}\sin\left(\frac{1}{2}\pi s\right)$$

Then you made the comment that if you take the limit of both sides as $s \to 0$, the $\zeta(1-s)$ term goes to infinity while the $\sin\left(\frac{1}{2}\pi s\right)$ term goes to zero in such a way that the whole expression goes to $-\frac{1}{2}$. This would then say that $\zeta(0) = -\frac{1}{2}$.

Let me now return to what I wrote on the blackboard that morning before math class started. You may already know this "derivation," but I'll present it anyways. On the blackboard that morning I said: "take the Laplace transform of $\sin(x)$:"

$$I(\alpha) = \int_0^{\infty} e^{-\alpha x}\sin(x)\,dx \quad \text{for } \alpha > 0$$

(We were in fact studying differential equations at the time) The fact that this was called a Laplace transform didn't bother me. I just wanted to construct this function $I(\alpha)$. Now we can evaluate the above integral by doing integration by parts two times (you can also do it easily by writing $\sin(x) = \frac{e^{ix}-e^{-ix}}{2i}$):

$$\int_0^\infty e^{-\alpha x}\sin(x)\,dx = -\frac{e^{-\alpha x}}{\alpha}\sin(x)\Big|_0^\infty + \frac{1}{\alpha}\int_0^\infty e^{-\alpha x}\cos(x)\,dx =$$

$$\frac{1}{\alpha}\left(-\frac{e^{-\alpha x}}{\alpha}\cos(x)\Big|_0^\infty - \frac{1}{\alpha}\int_0^\infty e^{-\alpha x}\sin(x)\,dx\right) = \frac{1}{\alpha^2} - \frac{1}{\alpha^2}\int_0^\infty e^{-\alpha x}\sin(x)\,dx$$

So:

$$\left(1+\frac{1}{\alpha^2}\right)\int_0^\infty e^{-\alpha x}\sin(x)\,dx = \frac{1}{\alpha^2}$$

$$\int_0^\infty e^{-\alpha x}\sin(x)\,dx = \frac{\frac{1}{\alpha^2}}{\left(1+\frac{1}{\alpha^2}\right)} = \frac{1}{1+\alpha^2}$$

And so we get:

$$\int_0^\infty e^{-\alpha x}\sin(x)\,dx = \frac{1}{1+\alpha^2}$$

Now, take the limit of both sides as $\alpha \to 0^+$. We get that:

$$\lim_{\alpha\to 0^+}\left(\int_0^\infty e^{-\alpha x}\sin(x)\,dx\right) = \int_0^\infty \lim_{\alpha\to 0^+}\left(e^{-\alpha x}\sin(x)\right)dx = \int_0^\infty \sin(x)\,dx$$

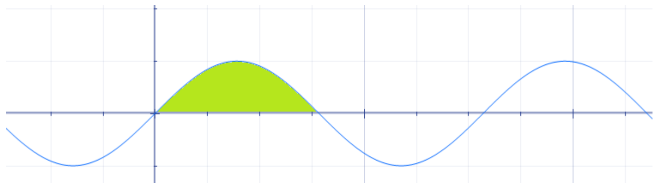
And on the other hand:

$$\lim_{\alpha\to 0^+}\left(\int_0^\infty e^{-\alpha x}\sin(x)\,dx\right) = \lim_{\alpha\to 0^+}\left(\frac{1}{1+\alpha^2}\right) = 1$$

So combining the two results give:

$$\int_0^\infty \sin(x)\,dx = 1$$

By this point the professor must have been thinking “What is this student thinking?! Bending and twisting the mathematics that I teach in this class in this manner.” Now $\sin(x)$ looks like (all of the graphs in this letter were made in Microsoft Mathematics):



And all freshman calculus students who have been doing their homework know that the absolute value of the area between each of the arcs and the x-axis (shown by the shaded region above) is 2. So we can write the integral in the last equation as:

$$\sum_{k=1}^{\infty}(-1)^k * 2 = 1$$

Dividing both sides by 2 will give the equation that I wrote on the blackboard:

$$\sum_{k=1}^{\infty}(-1)^k = \frac{1}{2}$$

But we won't stop here. On page 153 you mention the subject of analytic continuation of the zeta function. On that page you say that one of the formulas that mathematicians use to analytically extend the zeta function beyond where it is classically not defined, is the formula:

$$\sum_{k=1}^{\infty}\frac{(-1)^k}{k^z} = (1-2^{1-z})\zeta(z)$$

This formula is very easy to derive for z such that (remember z could be complex too) $Re(z) > 1$ (The region where $Re(z) > 1$ is in fact the region in the complex plain where the series $\sum_{k=1}^{\infty}\frac{1}{k^z}$ converges). The left hand side of the equation is in fact called Dirichlet's eta function, and is denoted by:

$$\eta(z) = \sum_{k=1}^{\infty}\frac{(-1)^k}{k^z}$$

So we have:

$$\eta(z) = (1-2^{1-z})\zeta(z)$$

Now looking back at the formula that I wrote on the blackboard, we see that:

$$\eta(0) = \sum_{k=1}^{\infty}(-1)^k = \frac{1}{2}$$

And so we have that:

$$\frac{1}{2} = (1-2^{1-0})\zeta(0)$$

Rearrangement finally gives us:

$$\zeta(0) = -\frac{1}{2}$$

The correct answer! Yeah, we're both thinking the same thing. Bold! And of course my use of the limit was absolutely wrong. To be specific, the step when the correctness of the derivation falls apart was when I did

$$\lim_{\alpha\to 0^+}\left(\int_0^\infty e^{-\alpha x}\sin(x)\,dx\right) = \int_0^\infty \lim_{\alpha\to 0^+}(e^{-\alpha x}\sin(x))\,dx =$$

But hey, it gave us the correct answer. Formal analytic continuation of the zeta function gives the same answer. I mean if my analysis teacher had the right to be bold using the Dirac delta function when solving differential equations (although I will admit that he wasn't being as bold as I am in this case), I have the right to be bold as well.

I didn't stop here. Let us try to compute the zeta function at further negative values. Here we take a different approach. You show in your book that $\zeta(-2n) = 0$ for positive integers n. We could in fact try to compute the zeta function at all negative integer values. Our calculation will again require us to be bold. You know, I don't know whether boldness was a good or bad trait that I picked up from you and Euler. Anyways, take the famous series expansion:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{k=0}^{\infty} x^k$$

This formula is in fact yet another source of the formula $\sum_{k=1}^{\infty}(-1)^k = \frac{1}{2}$. Just plug in -1 into x above. That should comfort us that there is some truth in all of this madness since two different methods yield the same answer. Anyways, let us differentiate both sides:

$$\frac{-1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots = \sum_{k=0}^{\infty}(k+1)x^k$$

Now take the limit of both sides as $x \to -1^+$.The left hand side converges to $-\frac{1}{4}$, and the right hand side converges to the series:

$$1 - 2 + 3 - 4 + 5 - 6 + 7 - 8 + \cdots = \sum_{k=0}^{\infty}(-1)^k(k+1)$$

So we have:

$$-\frac{1}{4} = 1 - 2 + 3 - 4 + 5 - 6 + 7 - 8 + \cdots = \sum_{k=0}^{\infty}(-1)^k(k+1)$$

Now notice that the right hand side is the Dirichlet eta function, evaluated at -1. So we get:

$$\eta(-1) = -\frac{1}{4}$$

Using the formula $\eta(z) = (1 - 2^{1-z})\zeta(z)$, we get that:

$$\left(1 - 2^{1-(-1)}\right)\zeta(-1) = -\frac{1}{4}$$

And rearranging finally gives:

$$\zeta(-1) = -\frac{1}{12}$$

This also agrees with the values of the zeta function when one does analytic continuation to -1. Pretty cool, huh! To get the next value $\zeta(-2)$, one just has to multiply the above Taylor polynomial by x and then differentiate again to get:

$$\frac{d}{dx}\left(x \cdot \frac{-1}{(1-x)^2}\right) = 1 + 2^2x + 3^2x^2 + 4^2x^3 + \cdots = \sum_{k=0}^{\infty}(k+1)^2x^k$$

Then take the limit of both sides as $x \to -1^+$. This gives:

$$\lim_{x\to-1^+}\left(\frac{d}{dx}\left(x \cdot \frac{-1}{(1-x)^2}\right)\right) = \lim_{x\to-1^+}\left(-\frac{1}{(1-x)^2} - \frac{2}{(1-x)^3}\right) = 0 =$$

$$\lim_{x\to-1^+}\left(\sum_{k=0}^{\infty}(k+1)^2x^k\right) = \sum_{k=0}^{\infty}(-1)^k(k+1)^2 = \eta(-2)$$

So:

$$\eta(-2) = 0$$

And so the formula $\eta(z) = (1 - 2^{1-z})\zeta(z)$ finally gives:

$$\zeta(-2) = 0$$

Just like you said it should be. To get further negative values of the zeta function, just repeat the same process over and over again in order to get that for any positive integer n:

$$\zeta(-n) = \frac{1}{(1-2^{1+n})} \cdot \lim_{x\to-1^+}\left(\frac{d}{dx}\left(x \cdot \frac{d}{dx}\left(x \cdot \frac{d}{dx}\left(x \cdot \ldots \cdot \frac{d}{dx}\left(\frac{1}{(1-x)}\right)\ldots\right)\right)\right)\right)$$

Wikipedia has the following formula for $\zeta(-n)$:

$$\zeta(-n) = -\frac{B_{n+1}}{n+1}$$

Where B_n denotes the n^th^ Bernoulli number. I naturally wanted to and have yet not been able to prove that:

$$-\frac{B_{n+1}}{n+1}=\frac{1}{(1-2^{1+n})}\cdot\lim_{x\to-1^+}\left(\frac{d}{dx}\left(x\cdot\frac{d}{dx}\left(x\cdot\frac{d}{dx}\left(x\cdot\ldots\cdot\frac{d}{dx}\left(\frac{1}{(1-x)}\right)\ldots\right)\right)\right)\right)$$

Related to the zeta function, Euler was able to prove that:

$$\zeta(2n)=\frac{(-1)^{n+1}B_{2n}(2\pi)^{2n}}{2(2n)!}$$

This is yet another formula that I have tried and have not been able to prove so far. But I will keep trying.

The Singularity of the Zeta Function at 1

In the last section in chapter 6 you derive the zeta functional equation. On pages 185 – 186 you say that in order to show that $\zeta(0)=-\frac{1}{2}$, one has to show that the singularity of the Riemann-Zeta function at 1 is powerful enough to overcome the zero of $\sin(x)$ at $x=0$ in order to get a finite non-zero limit. More rigorously one has to show that:

$$\lim_{x\to1^+}(\zeta(x)\sin(x-1))\text{ exists and is non-zero.}$$

Since $\lim_{x\to1^+}\left(\frac{\sin(x-1)}{x-1}\right)=1$, we can then rewrite the above limit as:

$$\lim_{x\to1^+}(\zeta(x)\sin(x-1))=\frac{\lim_{x\to1^+}(\zeta(x)\sin(x-1))}{\lim_{x\to1^+}\left(\frac{\sin(x-1)}{x-1}\right)}=\lim_{x\to1^+}\left(\frac{\zeta(x)\sin(x-1)}{\frac{\sin(x-1)}{x-1}}\right)=$$

$$\lim_{x\to1^+}\big(\zeta(x)(x-1)\big)$$

And so really one just needs to prove the following fact:

$$\lim_{x\to1^+}\big(\zeta(x)(x-1)\big)\text{ exists and is non-zero.}$$

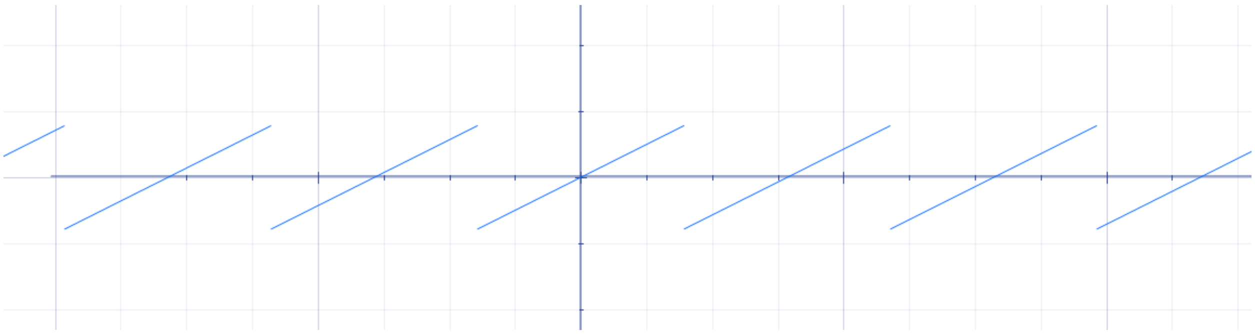
Let me discuss what I did in order to explore what kind of singularity the zeta function has at one.

In this section I will be skipping a lot of steps. The steps in this section are more for showing the outline of the derivation. So the derivation may be hard to follow on first glance.

One day I was sitting and I decided to explore something that one could call “hyper-Fourier series.” By this I meant Fourier series that had t^3 in the arguments of the trigonometric functions instead of just plain old t. Any undergraduate who has studied Fourier series knows the most famous of Fourier series: the saw tooth (here $\lfloor\quad\rfloor$ means the floor function):

$$\frac{x}{2} - \left\lfloor \frac{x+\pi}{2\pi} \right\rfloor \pi = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \sin(kx)$$

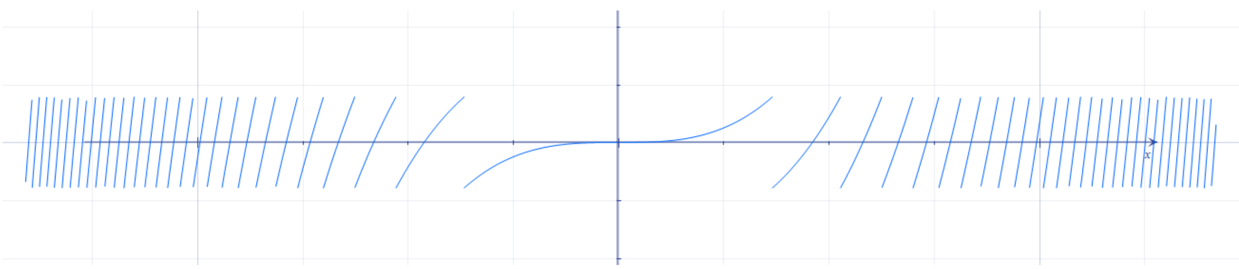
Which looks like:



Then I went ahead and plugged in t^3 into x on both sides of the equation, which will give us:

$$\frac{t^3}{2} - \left\lfloor \frac{t^3+\pi}{2\pi} \right\rfloor \pi = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \sin(kt^3)$$

If we plot the graph of this we get:



Then I decided to integrate both sides of the equation from 0 to ∞. Integrating the left side is a little bit tricky, but correctly parametrizing and integrating each piece (separated by jumps) and then adding up all of the results will give:

$$\int_0^{\infty} \left(\frac{t^3}{2} - \left\lfloor \frac{t^3+\pi}{2\pi} \right\rfloor \pi \right) dt = \frac{\pi^{\frac{4}{3}}}{8} + \frac{\pi^{\frac{4}{3}}}{8} \sum_{k=2}^{\infty} \left((6k-5)(2k-3)^{\frac{1}{3}} - (6k-7)(2k-1)^{\frac{1}{3}} \right)$$

Integrating the right hand side of the previous equation and interchanging the integral with the sum gives (just like you say in your book, I can interchange sums and integrals without hesitation and faster than anyone can snap their fingers – as long as I get the correct answer):

$$\int_0^{\infty} \left(\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \sin(kt^3) \right) dt = \sum_{k=1}^{\infty} \left(\frac{(-1)^{k+1}}{k} \int_0^{\infty} \sin(kt^3)\, dt \right)$$

Now using the fact that $\int_0^\infty \sin(t^n)\,dt = \sin\left(\frac{\pi}{2n}\right)\Gamma\left(\frac{n+1}{n}\right)$ – something that I will show you how to derive using complex analysis shortly in this letter – one gets that:

$$\sum_{k=1}^{\infty}\left(\frac{(-1)^{k+1}}{k}\int_0^\infty \sin(kt^3)\,dt\right) = \sin\left(\frac{\pi}{6}\right)\Gamma\left(\frac{4}{3}\right)\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^{\frac{4}{3}}} = \frac{2^{\frac{4}{3}}-2}{2^{\frac{4}{3}}}\sin\left(\frac{\pi}{6}\right)\Gamma\left(\frac{4}{3}\right)\zeta\left(\frac{4}{3}\right)$$

In the last equality I used the relationship:

$$\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^z} = \frac{2^z-2}{2^z}\zeta(z)$$

Which can be easily derived by just rearranging terms. The left hand side is in fact called Dirichlet's Eta function. So combining back the two sides gives us the pretty cool identity:

$$\frac{\pi^{\frac{4}{3}}}{8}+\frac{\pi^{\frac{4}{3}}}{8}\sum_{k=2}^{\infty}\left((6k-5)(2k-3)^{\frac{1}{3}}-(6k-7)(2k-1)^{\frac{1}{3}}\right) = \frac{2^{\frac{4}{3}}-2}{2^{\frac{4}{3}}}\sin\left(\frac{\pi}{6}\right)\Gamma\left(\frac{4}{3}\right)\zeta\left(\frac{4}{3}\right)$$

Or multiplying through by $\frac{8}{\pi^{\frac{4}{3}}}$ gives us:

$$1+\sum_{k=2}^{\infty}\left((6k-5)(2k-3)^{\frac{1}{3}}-(6k-7)(2k-1)^{\frac{1}{3}}\right) = \frac{8}{\pi^{\frac{4}{3}}}\cdot\frac{2^{\frac{4}{3}}-2}{2^{\frac{4}{3}}}\sin\left(\frac{\pi}{6}\right)\Gamma\left(\frac{4}{3}\right)\zeta\left(\frac{4}{3}\right)$$

An amazing looking formula! In fact one can check numerically that the left hand side converges to the quantity on the right, as it should.

Now remember how in an early step I plugged in t^3 into x? I could have plugged in t^n into x instead where n is an odd positive integer (I have a bad feeling about even n, so I didn't look at them) and then I would have arrived at the more general result:

$$1+\sum_{k=2}^{\infty}\left(\big(2nk-(2n-1)\big)(2k-3)^{\frac{1}{n}}-\big(2nk-(2n+1)\big)(2k-1)^{\frac{1}{n}}\right) =$$

$$\frac{2(n+1)}{\pi^{\frac{n+1}{n}}}\cdot\frac{2^{\frac{n+1}{n}}-2}{2^{\frac{n+1}{n}}}\sin\left(\frac{\pi}{2n}\right)\Gamma\left(\frac{n+1}{n}\right)\zeta\left(\frac{n+1}{n}\right)$$

This also is a remarkable looking formula! But the cool thing is, is that I can take this formula and take the limit of both sides as $n\to\infty$ in order to analyze the point of singularity of $\zeta(z)$ on the real axis as you approach 1 from the right side. Many of the terms on the right side will go away. For example $\Gamma\left(\frac{n+1}{n}\right)$ will tend to 1 as $n\to\infty$. $\sin\left(\frac{\pi}{2n}\right)$ will tend to zero approximately like $\frac{\pi}{2n}$ will (since $\lim_{n\to\infty}\left(\frac{\sin\left(\frac{\pi}{2n}\right)}{\frac{\pi}{2n}}\right) = 1$). So taking the limit of both sides as $n \to \infty$, rearranging terms, using L'Hopital's rule in the mix, and relabeling variables will give the result that:

$$\lim_{x\to 1^+}\big(\zeta(x)(x-1)\big) =$$

$$\frac{1}{\ln(2)}\cdot\lim_{n\to\infty}\left(1+\sum_{k=2}^{\infty}\left(\big(2nk-(2n-1)\big)(2k-3)^{\frac{1}{n}}-\big(2nk-(2n+1)\big)(2k-1)^{\frac{1}{n}}\right)\right)$$

I think that this is an incredible looking formula. But it was especially amazing to me because using inequalities and approximating integrals, I was able to prove that the right hand side of the above equation lies somewhere in the interval $[0.5, \frac{1}{\ln(2)}]$ (I will not burden you with those details). Thus I was able to prove that as the Riemann Zeta function approaches 1 from the right on the real axis, it will have a singularity of order 1 there:

$$\lim_{x\to 1^+}\big(\zeta(x)(x-1)\big) \quad \textit{is somewhere in the interval} \quad [0.5, \frac{1}{\ln(2)}]$$

This proves what we wanted.

Later I talked to my first year honors calculus teacher at the University of Washington and he showed me how to exactly compute the above limit using an analytic extension of the Riemann-Zeta function. Precisely he showed me how to prove that:

$$\lim_{z\to 1}\big(\zeta(z)(z-1)\big) = 1$$

Where z is a complex variable that can approach 1 from any direction and on any path that converges to 1. I decided to leave the proof of the above equation out of this letter, but not to leave you unsatisfied here, combining his and my result (and just a little bit of algebra) leads to the following formula:

$$\lim_{n\to\infty}\left(n\sum_{k=2}^{\infty}\left(\left(2k-2-\frac{1}{n}\right)(2k-1)^{\frac{1}{n}}-\left(2k-2+\frac{1}{n}\right)(2k-3)^{\frac{1}{n}}\right)\right) = 1-\ln(2)$$

A result that one can truly be proud of.

Complex Analysis

Subsection 1: The Hyper-Fresnel Integrals:

There will be 3 "subsections" in the section "Complex Analysis" since this section is so large.

Now we get to my currently favorite subject of mathematics: complex analysis. Although I will point out that I don't bias against any field of mathematics when I do personal research. What I mean is that when I say "my favorite math subject", well… that doesn't mean much. This chapter was the most astounding of them all! Chapter 6 was amazing, but this one was even greater in my opinion. In fact nearly everything I know about complex integration comes from this book. From the period after I read this chapter, I've learned a lot more about complex integration from discovering facts for myself or little independent readings from various sources. But the basis of my knowledge comes from this book. As I'm writing this section, I don't even know where I should begin.

In your section on complex analysis you derive the Fresnel integrals (I was not satisfied with your derivation of the Fresnel integrals in chapter 6 though, so I consider the true derivation to be the one in the complex analysis section). These integrals are fantastic! I remember that when I showed these integrals to my math professor for the first time, the first thing he told me was "these integrals don't even converge." Later a little thought convinced him that these integrals did in fact converge. See how deceptive these integrals are?

One of the things that one might wonder is why the two integrals must be equivalent:

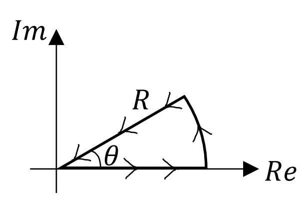
$$\int_0^\infty \cos(x^2)\,dx = \int_0^\infty \sin(x^2)\,dx$$

Another thing that someone might wonder is: can one generalize the Fresnel integrals any further? The answer to both questions are the equations:

$$\int_0^\infty \sin(x^n)\,dx = \sin\left(\frac{\pi}{2n}\right)\Gamma\left(\frac{n+1}{n}\right)$$

$$\int_0^\infty \cos(x^n)\,dx = \cos\left(\frac{\pi}{2n}\right)\Gamma\left(\frac{n+1}{n}\right)$$

Where $\Gamma(x)$ is the gamma function. The above formulas are important because they provide a good method for numerically calculating these integrals. Can you imagine how one would go about calculating numerically the above integrals? Direct numerical computation would converge extremely slowly for large n. But calculating $\sin\left(\frac{\pi}{2n}\right)\Gamma\left(\frac{n+1}{n}\right)$ and $\cos\left(\frac{\pi}{2n}\right)\Gamma\left(\frac{n+1}{n}\right)$ are easy. The way I derived the above integrals was to just generalize the calculation you did on pages 199 – 201. All I did was, instead of integrating the complex function e^{-z^2}, I integrated e^{-z^n}. The contour that I integrated on was the following pie shaped slice in the complex plain:



Let the angle of the pie slice be $\theta = \frac{\pi}{2n}$. R is the radius of the pie slice. Let us call this contour upon which we integrate γ (Technically γ should be a function of R because we are later going to stretch the contour by letting $R \to \infty$. So we should be writing γ_R. But for brevity I will drop the subscript and just write γ). Now since e^{-z^n} is an analytic function and γ is a closed looped contour, we have that:

$$\oint_{\gamma} e^{-z^n} dz = 0$$

Now let us expand the integral:

$$\oint_{\gamma} e^{-z^n} dz = \int_0^R e^{-x^n} dx + i \int_0^{\frac{\pi}{2n}} e^{-R^n e^{in\theta}} Re^{i\theta} d\theta$$
$$- \int_0^R e^{-\left(\cos\left(\frac{\pi}{2n}\right)+i\sin\left(\frac{\pi}{2n}\right)\right)^n x^n} \left(\cos\left(\frac{\pi}{2n}\right) + i\sin\left(\frac{\pi}{2n}\right)\right) dx$$

By DeMoivre's Formula, we know that the quantity in the exponent in the last integral is:

$$\left(\cos\left(\frac{\pi}{2n}\right) + i\sin\left(\frac{\pi}{2n}\right)\right)^n = \cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right) = i$$

So we have that:

$$\oint_{\gamma} e^{-z^n} dz = \int_0^R e^{-x^n} dx + i \int_0^{\frac{\pi}{2n}} e^{-R^n e^{in\theta}} Re^{i\theta} d\theta - \int_0^R e^{-ix^n} \left(\cos\left(\frac{\pi}{2n}\right) + i\sin\left(\frac{\pi}{2n}\right)\right) dx$$

Now take the limit of both sides as $R \to \infty$ (remember, the left hand side is constantly zero and so its limit is zero) and get that:

$$\int_0^\infty e^{-x^n} dx + \lim_{R\to\infty}\left(i \int_0^{\frac{\pi}{2n}} e^{-R^n e^{in\theta}} Re^{i\theta} d\theta \right) - \int_0^\infty e^{-ix^n} \left(\cos\left(\frac{\pi}{2n}\right) + i\sin\left(\frac{\pi}{2n}\right)\right) dx = 0$$

It isn't hard to show (following the same logic as in your book) that:

$$\lim_{R\to\infty}\left(i \int_0^{\frac{\pi}{2n}} e^{-R^n e^{in\theta}} Re^{i\theta} d\theta \right) = 0$$

So we get that:

$$\int_0^\infty e^{-x^n}dx = \int_0^\infty e^{-ix^n}\left(\cos\left(\frac{\pi}{2n}\right) + i\sin\left(\frac{\pi}{2n}\right)\right)dx$$

Now, let us handle the right hand side of this equation first:

$$\int_0^\infty e^{-ix^n}\left(\cos\left(\frac{\pi}{2n}\right) + i\sin\left(\frac{\pi}{2n}\right)\right)dx = \int_0^\infty (\cos(x^n) - i\sin(x^n))\left(\cos\left(\frac{\pi}{2n}\right) + i\sin\left(\frac{\pi}{2n}\right)\right)dx =$$

$$\left(\cos\left(\frac{\pi}{2n}\right)\int_0^\infty \cos(x^n)\,dx + \sin\left(\frac{\pi}{2n}\right)\int_0^\infty \sin(x^n)\,dx\right)$$
$$+ i\left(\sin\left(\frac{\pi}{2n}\right)\int_0^\infty \cos(x^n)\,dx - \cos\left(\frac{\pi}{2n}\right)\int_0^\infty \sin(x^n)\,dx\right)$$

Notice that this expression has to be real because the integral $\int_0^\infty e^{-x^n}dx$ (the left hand side of the previous equation) is a real number. This then means two things:

$$\int_0^\infty e^{-x^n}dx = \cos\left(\frac{\pi}{2n}\right)\int_0^\infty \cos(x^n)\,dx + \sin\left(\frac{\pi}{2n}\right)\int_0^\infty \sin(x^n)\,dx$$

And:

$$\sin\left(\frac{\pi}{2n}\right)\int_0^\infty \cos(x^n)\,dx - \cos\left(\frac{\pi}{2n}\right)\int_0^\infty \sin(x^n)\,dx = 0$$

The second equation can be rewritten as:

$$\int_0^\infty \cos(x^n)\,dx = \cot\left(\frac{\pi}{2n}\right)\int_0^\infty \sin(x^n)\,dx$$

Plugging this into the first equation gives us that:

$$\int_0^\infty e^{-x^n}dx = \left(\cos\left(\frac{\pi}{2n}\right)\cot\left(\frac{\pi}{2n}\right) + \sin\left(\frac{\pi}{2n}\right)\right)\int_0^\infty \sin(x^n)\,dx$$

And further simplification gives:

$$\int_0^\infty \sin(x^n)\,dx = \sin\left(\frac{\pi}{2n}\right)\int_0^\infty e^{-x^n}dx$$

Amazing already! Now the only part that remains to show is that the integral $\int_0^\infty e^{-x^n}dx$ is connected to the Gamma Function. But that's easy! From the definition of the Gamma function we know that:

$$\Gamma\left(\frac{1}{n}\right) = \int_0^\infty x^{\frac{1}{n}-1}e^{-x}dx = \int_0^\infty \frac{e^{-x}}{x^{\frac{n-1}{n}}}dx$$

Making the substitution $x = u^n$, one will that get (remember, this gives that $dx = n \cdot u^{n-1}du$)

$$\Gamma\left(\frac{1}{n}\right) = \int_0^\infty \frac{e^{-u^n}}{u^{n-1}}n \cdot u^{n-1}du = n\int_0^\infty e^{-u^n}du$$

So we get that:

$$\int_0^\infty e^{-x^n}dx = \frac{1}{n}\Gamma\left(\frac{1}{n}\right) = \Gamma\left(\frac{n+1}{n}\right)$$

Substituting this into the integral $\int_0^\infty e^{-x^n}dx$ a couple of equations back will finally give us what we want:

$$\int_0^\infty \sin(x^n)\,dx = \sin\left(\frac{\pi}{2n}\right)\Gamma\left(\frac{1}{n}\right)$$

And if a couple of equations back we would have solved for $\int_0^\infty \cos(x^n)\,dx$ instead of $\int_0^\infty \sin(x^n)\,dx$, we would then have gotten the other result:

$$\int_0^\infty \cos(x^n)\,dx = \cos\left(\frac{\pi}{2n}\right)\Gamma\left(\frac{1}{n}\right)$$

Putting them together we get the amazing pair of equations side by side:

$$\int_0^\infty \sin(x^n)\,dx = \sin\left(\frac{\pi}{2n}\right)\Gamma\left(\frac{n+1}{n}\right)$$

$$\int_0^\infty \cos(x^n)\,dx = \cos\left(\frac{\pi}{2n}\right)\Gamma\left(\frac{n+1}{n}\right)$$

Amazing right! I really don't know how I would have gone about proving these results without the use of complex analysis. If I were to give these integrals a name, I would call them Hyper-Fresnel integrals – you know, because they look like the Fresnel integrals but they are "hyper" since they instead involve raising x to the n^{th} power.

Subsection 2: Extension of a Really Important Integral:

In your derivation of the gamma reflection formula you use the very important integral:

$$\int_0^\infty \frac{x^{\alpha-1}}{1+x^\beta}dx = \frac{\pi}{\beta\sin\left(\frac{\alpha}{\beta}\pi\right)}$$

In chapter 6 you state that you will prove this integral in chapter 7, and indeed you do prove it there using complex analysis. However your derivation in chapter 7 only covers the case when α is an integer of the form $\alpha = 2m+1$ where m is a nonnegative integer and β is of the form $\beta = 2n$ where n is a positive integer such that $n > m$. I later started to think about how could one extend this equation to positive real numbers α and β. Here I will show an approach on how to do that.

Let us take positive real numbers $\alpha - 1$ and β such that $\beta > \alpha - 1$. Let us take sequences of rational numbers $\left\{\frac{a_n}{b_n}\right\}_{n=1}^{\infty}$ and $\left\{\frac{p_n}{q_n}\right\}_{n=1}^{\infty}$ that converge to $(\alpha - 1)$ and β respectively. In other words let:

$$\left\{\frac{a_n}{b_n}\right\} \to \alpha - 1 \quad and \quad \left\{\frac{p_n}{q_n}\right\} \to \beta \quad as \quad n \to \infty$$

Let us agree that all of the rational numbers in these sequences are in simplified reduced form. So here we have that for increasingly big n, $\frac{a_n}{b_n}$ and $\frac{p_n}{q_n}$ better and better approximate $\alpha - 1$ and β resepectivly. In fact one of $\frac{a_n}{b_n}$ or $\frac{p_n}{q_n}$ may be equal to $\alpha - 1$ or β respectively if either $\alpha - 1$ or β happens to be rational. Point being, $\frac{a_n}{b_n}$ and $\frac{p_n}{q_n}$ arbitrarily well approximate $\alpha - 1$ and β respectively as n increases to ∞. So let us plug in $\frac{a_n}{b_n}$ and $\frac{p_n}{q_n}$ into $\alpha - 1$ and β respectively in the above integral:

$$\int_0^\infty \frac{x^{\frac{a_n}{b_n}}}{1+x^{\frac{p_n}{q_n}}}dx$$

Now let us do some algebra and substitutions:

$$\int_0^\infty \frac{x^{\frac{a_n}{b_n}}}{1+x^{\frac{p_n}{q_n}}}dx = \int_0^\infty \frac{x^{\frac{a_n \cdot q_n}{b_n \cdot q_n}}}{1+x^{\frac{p_n \cdot b_n}{q_n \cdot b_n}}}dx$$

Now let us make the substitution $u = x^{\frac{1}{b_n \cdot q_n}}$. With this substitution we get that:

$$u^{b_n q_n} = x$$

So:

$$b_n q_n u^{b_n q_n - 1} du = dx$$

And so the above integral becomes:

$$b_n q_n \int_0^\infty \frac{u^{a_n q_n + b_n q_n - 1}}{1 + u^{p_n b_n}} dx = b_n q_n \int_0^\infty \frac{u^{q_n(a_n + b_n) - 1}}{1 + u^{p_n b_n}} dx$$

Now let us go back and agree that the sequences $\left\{\frac{a_n}{b_n}\right\}$ and $\left\{\frac{p_n}{q_n}\right\}$ include only rational numbers where q_n and b_n are divisible by two. This is an ok restriction to do because you can still always pick such rational numbers that still converge to $\alpha - 1$ and β respectively. With this in hand, we then know that a_n and p_n cannot be divisible by two (or else at least one of the fractions $\frac{a_n}{b_n}$ or $\frac{p_n}{q_n}$ would be reducible and we agreed before that the fractions in the sequences $\left\{\frac{a_n}{b_n}\right\}$ and $\left\{\frac{p_n}{q_n}\right\}$ are all in simplified reduced form). Thus the quantities $q_n(a_n + b_n) - 1$ and $p_n b_n$ are both divisible by two. So let us write these numbers as:

$$q_n(a_n + b_n) - 1 = 2m$$

$$p_n b_n = 2s$$

For some positive integers m and s. So now we can write the last integral as:

$$b_n q_n \int_0^\infty \frac{u^{2m}}{1 + u^{2s}} dx$$

But in fact, you take this last integral in your book on pages 213-217. So we get that:

$$b_n q_n \int_0^\infty \frac{u^{2m}}{1 + u^{2s}} dx = b_n q_n \cdot \frac{\pi}{2s \cdot \sin\left(\frac{2m+1}{2s}\pi\right)} = b_n q_n \cdot \frac{\pi}{p_n b_n \cdot \sin\left(\frac{q_n(a_n + b_n)}{p_n b_n}\pi\right)} =$$

$$\frac{\pi}{\frac{p_n}{q_n} \cdot \sin\left(\frac{\frac{a_n}{b_n} + 1}{\frac{p_n}{q_n}}\pi\right)}$$

So in the end we have:

$$\int_0^\infty \frac{x^{\frac{a_n}{b_n}}}{1 + x^{\frac{p_n}{q_n}}} dx = \frac{\pi}{\frac{p_n}{q_n} \cdot \sin\left(\frac{\frac{a_n}{b_n} + 1}{\frac{p_n}{q_n}}\pi\right)}$$

Now take the limit of both sides!

$$\lim_{n\to\infty}\left(\int_0^\infty \frac{x^{\frac{a_n}{b_n}}}{1+x^{\frac{p_n}{q_n}}}dx\right) = \int_0^\infty \frac{x^{\alpha-1}}{1+x^\beta}dx = \lim_{n\to\infty}\left(\frac{\pi}{\frac{p_n}{q_n}\cdot\sin\left(\frac{\frac{a_n}{b_n}+1}{\frac{p_n}{q_n}}\pi\right)}\right) = \frac{\pi}{\beta\sin\left(\frac{\alpha}{\beta}\pi\right)}$$

And so we get the final result that we wanted:

$$\int_0^\infty \frac{x^{\alpha-1}}{1+x^\beta}dx = \frac{\pi}{\beta\sin\left(\frac{\alpha}{\beta}\pi\right)}$$

Wow, right? So now we can say that we know how to prove this integral formula for all suitable positive real values $\alpha - 1$ and β.

Now that we have done this, we have to go back to some of the mathematical injustices. First of all we have to justify the operation:

$$\lim_{n\to\infty}\left(\int_0^\infty \frac{x^{\frac{a_n}{b_n}}}{1+x^{\frac{p_n}{q_n}}}dx\right) = \int_0^\infty \frac{x^{\alpha-1}}{1+x^\beta}dx$$

How do we know that limits interchange (integrals are limits too)? It would be enough to show that the integral $\int_0^\infty \frac{x^{\alpha-1}}{1+x^\beta}dx$ is a continuous function of both the variables α and β. I don't know how to do that. I heard that you can interchange derivatives with integrals and that differentiable functions are continuous. Maybe that would be a way to prove this. I don't know. I in fact don't know how to justify the above operation. Hey, I am only a student and not a professional mathematician. It is not always within my powers to justify everything rigorously. But what I can always do is explore, especially with the aid of computer computation. And I have explored and have very well convinced myself that the above integral equation is indeed true for all suitable positive real values $\alpha - 1$ and β.

Subsection 3: The Celestial Integral:

I did want to make one comment about the integral that you take in section 7.7 on page 218. In chapter 5 you show how this integral is connected with proving Kepler's 3rd law. In section 7.7 you compute this integral using complex analysis:

$$\int_0^{2\pi} \frac{1}{(1+E\cos(\theta))^2}d\theta = \frac{2\pi}{(1-E^2)^{\frac{3}{2}}} \quad for \quad 0 \le E < 1$$

I remember that when I first read your derivation of the above formula I was amazed at how well complex analysis is able to handle these types of integrals. I was also amazed by the generalization of Cauchy second integral theorem:

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_{\gamma} \frac{f(z)}{(z - z_0)^{n+1}} dz$$

I remember that I later sat down and took the integral by just substituting $E\cos(\theta)$ into x in the Taylor series expansion of $\frac{1}{(1+x)^2}$ (The following details are just supposed to be an illustration of the calculation, and are not in presentation mode. Meaning I skip steps here and so these details are not meant to be followed, at least easily that is):

$$\frac{1}{(1+x)^2} = 1 - 2x + 3x^2 - 4x^3 + 5x^4 - 6x^5 + \cdots = \sum_{k=0}^{\infty} kx^{k-1}$$

$$\frac{1}{(1 + E\cos(\theta))^2} = 1 - 2\cos(\theta) + 3\cos^2(\theta) - 4\cos^3(\theta) + 5\cos^4(\theta) - \cdots = \sum_{k=1}^{\infty} k\cos^{k-1}(\theta)$$

Now integrating both sides from 0 to 2π and using the fact that:

$$\int_0^{2\pi} \cos^n(x)\,dx = \begin{cases} 0 \quad if \quad n \equiv 1\ (mod\ 2) \\ \dfrac{\prod_{k=1}^{\frac{n}{2}} 2k - 1}{\prod_{k=1}^{\frac{n}{2}} 2k} 2\pi \quad if \quad n \equiv 0\ (mod\ 2) \end{cases}$$

One will get that:

$$\int_0^{2\pi} \frac{1}{(1 + E\cos(\theta))^2} d\theta = 2\pi\left(1 + \sum_{n=1}^{\infty}\left((2n+1)E^{2n}\frac{\prod_{k=1}^{n} 2k - 1}{\prod_{k=1}^{n} 2k}\right)\right) =$$

$$2\pi \sum_{n=0}^{\infty}\left(E^{2n}\frac{(2n+1)!}{2^{2n}(n!)^2}\right)$$

Now using the binomial expansion of $(1+x)^\alpha$ for real value α (This is just its Taylor series expansion around $x = 0$) we know that:

$$(1+x)^\alpha = 1 + \sum_{n=0}^{\infty}\left(\left(\prod_{k=1}^{n} \alpha - (k-1)\right)x^n\right)$$

One can notice that:

$$(1 - x^2)^{-\frac{3}{2}} = \sum_{n=0}^{\infty}\left(x^{2n}\frac{(2n+1)!}{2^{2n}(n!)^2}\right)$$

And so plugging in E into x in the above equation gives us that:

$$2\pi \sum_{n=0}^{\infty} \left(E^{2n} \frac{(2n+1)!}{2^{2n}(n!)^2} \right) = \frac{2\pi}{(1-E^2)^{-\frac{3}{2}}}$$

And so I finally got the result:

$$\int_0^{2\pi} \frac{1}{(1+E\cos(\theta))^2} d\theta = \frac{2\pi}{(1-E^2)^{\frac{3}{2}}}$$

The value of the integral $\int_0^{2\pi} \cos^n(x)\, dx$ that I wrote out above is in fact the connection between my derivation here and yours using complex analysis. In fact, the value of the integral is also deeply connected with Fourier series, thus hinting at some underlying connection between Fourier series and complex analysis, a connection that I will talk about later.

Part II:

When I finished the first book "An Imaginary Tale: The Story of $\sqrt{-1}$," I immediately wanted to get the second book "Dr. Euler's Fabulous Formula." Once I got my hands on it I wasted no time and started to read it (The page numbers below refer to the hard cover edition of "Dr. Euler's Fabulous Formula" that was published in 2006).

Ramanujan sum

When you look on page 28 at the series that Ramanujan summed we can immediately see that it is a Fourier series:

$$P(x) = \sum_{n=1}^{\infty} \frac{(-1)^n \cos(nx)}{(n+1)(n+2)}$$

Once I read about how Ramanujan summed this series I wanted to see if I can use his method in order to discover new things. Indeed I did! First of all I set out to derive the above sum in a more generalized form. Specifically I calculated the sum:

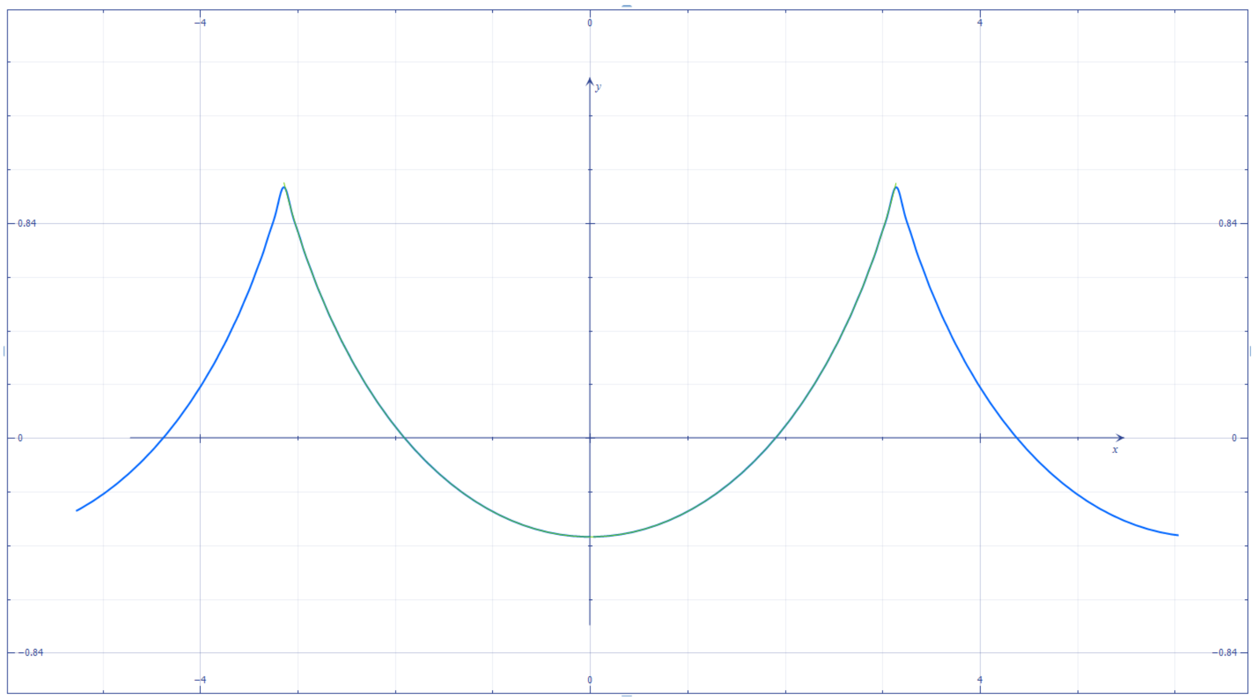
$$P_{m,n}(x) = \sum_{k=1}^{\infty} \frac{(-1)^k \cos(kx)}{(k+n)(k+m)}$$

Where m and n are two non-equal integers. I got (by just following the same exact steps as Ramanujan did) the result:

$$P_{m,n}(x) = \frac{1}{m-n}\left(\ln\left(2\cos\left(\frac{x}{2}\right)\right)[(-1)^m\cos(mx) - (-1)^n\cos(nx)] + \frac{x}{2}[(-1)^m\sin(mx) - (-1)^n\sin(nx)] + (-1)^n\sum_{k=1}^{n}\frac{(-1)^{k+1}\cos\big((n-k)x\big)}{k} - (-1)^m\sum_{k=1}^{m}\frac{(-1)^{k+1}\cos\big((m-k)x\big)}{k}\right)$$

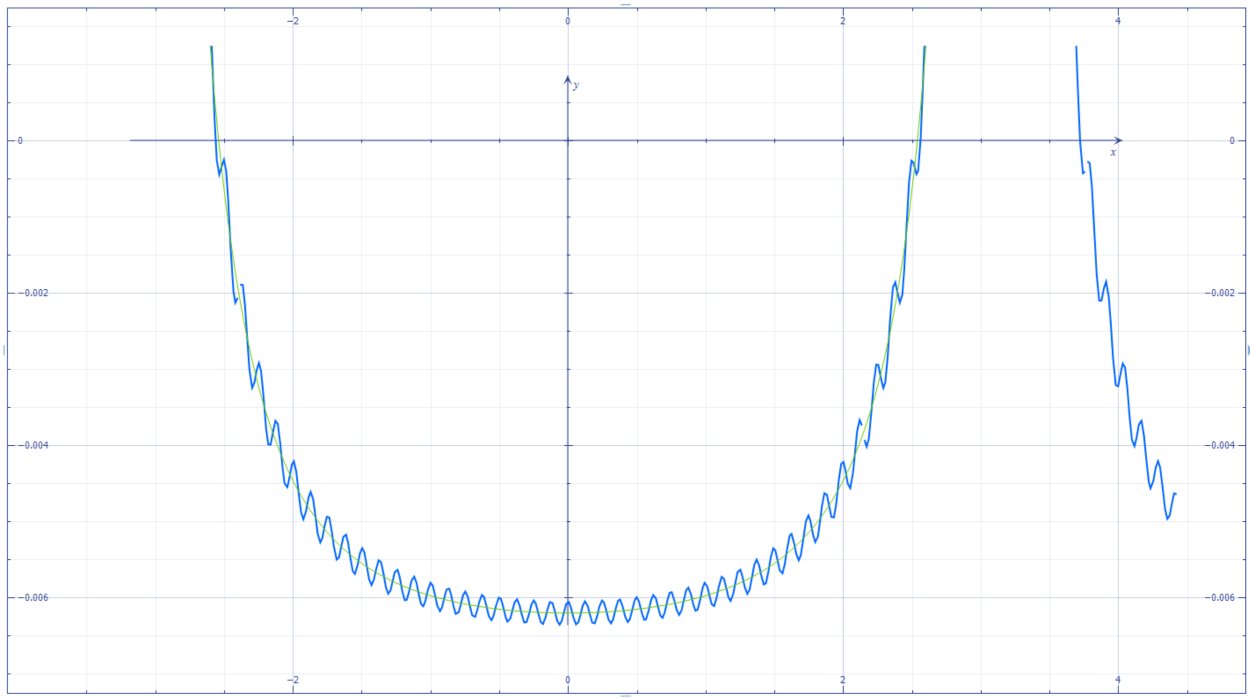
I personally think that it's really interesting to look at some graphs of these functions. So let us do exactly that.

Taking $n = 0$ and $m = 1$ we get the graph:

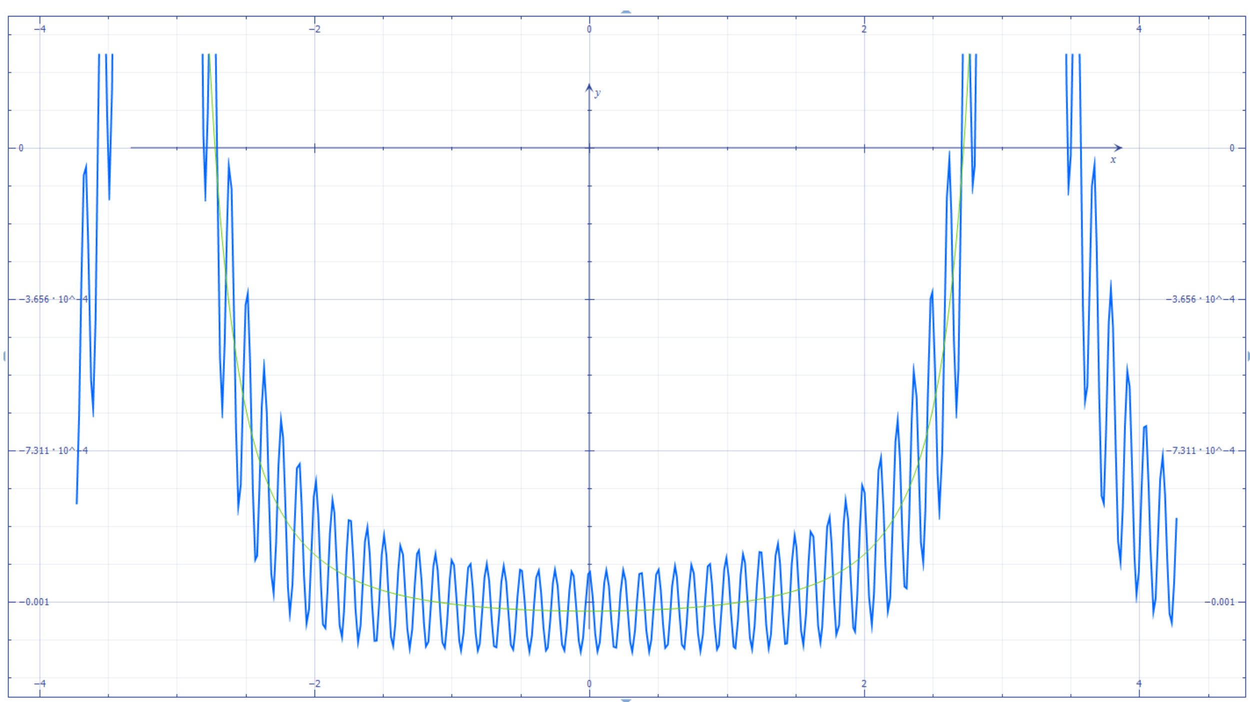


The blue curve is the series of $P_{1,0}(x)$ added up to 50 terms. The green curve is the exact evaluated answer for this series that is presented in the last equation. The fact that it is hard to distinguish the two graphs is due to the amazing agreement of the two results!

Now let us look at the case when $n = 7$ and $m = 11$:



Again here the series for $P_{7,11}(x)$ is added up to 50 terms. Finally let us look at the case when $n = 17$ and $m = 25$:



As before the series for $P_{17,25}(x)$ is added up to 50 terms. One of the things that I find fascinating and interesting in this sequence of graphs is that the amount of wiggling does not at all decrease as the integers m and n grow.

After this I was interested in whether I could derive some more interesting results by integrating both sides of the equation for $P_{m,n}(x)$. This is exactly what I did and let us see what kind of things I discovered in the next section.

Integrating Both Sides of the Ramanujan Sum Equation

In this section I analyze a special case, namely $P_{0,2}(x)$. I started off with:

$$P_{0,2}(x) = \sum_{k=1}^{\infty} \frac{(-1)^k \cos(kx)}{k(k+2)} = \frac{1}{2}\ln\left(2\cos\left(\frac{x}{2}\right)\right)[\cos(2x) - 1] + \frac{x}{4}\sin(2x) - \frac{1}{2}\cos(x) + \frac{1}{4}$$

Now let us integrate both sides from 0 to $\pi/2$. Integrating the left side of this equation gives us:

$$\int_0^{\frac{\pi}{2}} \sum_{k=1}^{\infty} \frac{(-1)^k \cos(kx)}{k(k+2)} dx = \sum_{k=1}^{\infty} \int_0^{\frac{\pi}{2}} \frac{(-1)^k \cos(kx)}{k(k+2)} dx = \left(\sum_{k=1}^{\infty} \frac{(-1)^k \sin(kx)}{k^2(k+2)}\right)\Bigg|_0^{\frac{\pi}{2}} =$$

$$\sum_{k=1}^{\infty} \frac{(-1)^k \sin\left(\frac{k\pi}{2}\right)}{k^2(k+2)} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2(2k+3)}$$

Integrating the right hand side of the equation gives us:

$$\frac{1}{2}\int_0^{\frac{\pi}{2}} \ln\left(2\cos\left(\frac{x}{2}\right)\right) \cdot \cos(2x)\, dx - \frac{1}{2}\int_0^{\frac{\pi}{2}} \ln\left(2\cos\left(\frac{x}{2}\right)\right) dx + \int_0^{\frac{\pi}{2}} \left(\frac{x}{4}\sin(2x) - \frac{1}{2}\cos(x) + \frac{1}{4}\right) dx =$$

$$\frac{1}{2}\int_0^{\frac{\pi}{2}} \ln\left(2\cos\left(\frac{x}{2}\right)\right) \cdot \cos(2x)\, dx - \frac{1}{2}\int_0^{\frac{\pi}{2}} \ln\left(2\cos\left(\frac{x}{2}\right)\right) dx + \frac{3\pi}{16} - \frac{1}{2}$$

Now the first two integrals in the above expression can be computed separately in pretty much exactly the same manner as you compute the integral $\int_0^{\pi/2} \ln(2\cos(x))\, dx$ on pages 32 – 33. The results that you will get for these integrals are (I won't bore you with the details of the actual calculations):

$$\int_0^{\frac{\pi}{2}} \ln\left(2\cos\left(\frac{x}{2}\right)\right) \cdot \cos(2x)\, dx = \frac{4-\pi}{8}$$

and

$$\int_0^{\frac{\pi}{2}} \ln\left(2\cos\left(\frac{x}{2}\right)\right) dx = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} = K$$

The last constant K is called Catalan's constant and it is just like $\zeta(3)$ in the fact that no one knows whether there is a closed form expression that is equal to it. Combing the two sides of the equation back finally gives us:

$$\sum_{k=1}^{\infty}\frac{(-1)^k}{(2k+1)^2(2k+3)}=\frac{1}{2}\left(\frac{4-\pi}{8}\right)-\frac{1}{2}K+\frac{3\pi}{16}-\frac{1}{2}=\frac{\pi}{8}-\frac{1}{4}-\frac{1}{2}K$$

And so:

$$\sum_{k=1}^{\infty}\frac{(-1)^k}{(2k+1)^2(2k+3)}=\frac{\pi-2}{8}-\frac{1}{2}K$$

A pretty result that can also be derived by separation of fractions.

The jewel of the above calculation however that I was super excited about was the discovery of the following fact:

$$K=\int_0^{\frac{\pi}{2}}\ln\left(2\cos\left(\frac{x}{2}\right)\right)dx$$

I was excited about this because I have for a long time have been trying to find the closed form expression for Catalan's constant. This isn't progress, but it is still a pretty cool result. In fact if we do the substitution $u=\frac{x}{2}$ we get that:

$$K=\int_0^{\frac{\pi}{2}}\ln\left(2\cos\left(\frac{x}{2}\right)\right)dx=2\int_0^{\frac{\pi}{4}}\ln(2\cos(u))\,du=2\int_0^{\frac{\pi}{4}}\ln(2)\,du+2\int_0^{\frac{\pi}{4}}\ln(\cos(u))\,du=$$

$$K=\frac{\pi}{2}\ln(2)+2\int_0^{\frac{\pi}{4}}\ln(\cos(u))\,du$$

A little bit cleaner formula for K in my opinion. It's interesting to note that this integral has a striking resemblance to the last integral on page 33 of your book. Is there a deep connection somewhere?

It was actually at this point that I realized immediately how to prove the formula that you present on page 149 in "An Imaginary Tale: The Story of $\sqrt{-1}$:"

$$\frac{1}{1^3}+\frac{1}{3^3}+\frac{1}{5^3}+\cdots=\frac{\pi^2}{4}\ln(2)+2\int_0^{\frac{\pi}{2}}x\ln(\sin(x))\,dx$$

It is easy to show that $\frac{1}{1^3}+\frac{1}{3^3}+\frac{1}{5^3}+\cdots=\frac{7}{8}\zeta(3)$, so the above equation really is equivalent to:

$$\zeta(3) = \frac{2\pi^2}{7}\ln(2) + \frac{16}{7}\int_0^{\frac{\pi}{2}} x\ln(\sin(x))\,dx$$

The proof of this result is really similar to the proof of the integral above that is equal to Catalan's constant. The result comes out pretty quickly and so let me show you how it's done in the next section.

Proof of $\frac{1}{1^3}+\frac{1}{3^3}+\frac{1}{5^3}+\cdots=\frac{\pi^2}{4}\ln(2)+2\int_0^{\frac{\pi}{2}} x\ln(\sin(x))\,dx$

In order to prove this equation we start off by evaluating the following integral:

$$\int_0^{\frac{\pi}{2}} x\ln(\sin(x))\,dx$$

We apply the same trick that you did on page 32 of your book "Dr. Euler's Fabulous Formula" to this integral. Namely we write:

$$\int_0^{\frac{\pi}{2}} x\ln(\sin(x))\,dx = \int_0^{\frac{\pi}{2}} x\ln\left(\frac{e^{ix}-e^{-ix}}{2i}\right)dx = \int_0^{\frac{\pi}{2}} x\left(\ln\left(e^{ix}-e^{-ix}\right)-\ln(2i)\right)dx =$$

$$\int_0^{\frac{\pi}{2}} x\ln\left(e^{ix}-e^{-ix}\right)dx - \ln(2i)\int_0^{\frac{\pi}{2}} x\,dx = \int_0^{\frac{\pi}{2}} x\ln\left(e^{ix}-e^{-ix}\right)dx - (\ln(2)+\ln(i))\frac{\pi^2}{8}$$

Using the fact that $\ln(i) = \frac{\pi}{2}i$, we get that:

Equation 1:
$$\int_0^{\frac{\pi}{2}} x\ln(\sin(x))\,dx = \int_0^{\frac{\pi}{2}} x\ln\left(e^{ix}-e^{-ix}\right)dx - \left(\ln(2)+\frac{\pi}{2}i\right)\frac{\pi^2}{8}$$

Let us compute the integral on the right hand side separately:

$$\int_0^{\frac{\pi}{2}} x\ln\left(e^{ix}-e^{-ix}\right)dx = \int_0^{\frac{\pi}{2}} x\ln\left(e^{ix}\left(1-e^{-2ix}\right)\right)dx = \int_0^{\frac{\pi}{2}} x\ln\left(e^{ix}\right)+x\ln\left(1-e^{-2ix}\right)dx =$$

$$\int_0^{\frac{\pi}{2}} ix^2\,dx + \int_0^{\frac{\pi}{2}} x\ln\left(1-e^{-2ix}\right)dx = i\frac{\pi^3}{24} + \int_0^{\frac{\pi}{2}} x\ln\left(1-e^{-2ix}\right)dx$$

So we have that:

Equation 2: $$\int_0^{\frac{\pi}{2}} x \ln\left(e^{ix} - e^{-ix}\right) dx = i\frac{\pi^3}{24} + \int_0^{\frac{\pi}{2}} x \ln\left(1 - e^{-2ix}\right) dx$$

Let us now compute the integral on the right side of this equation separately (Here I will use the Taylor series expansion of $\ln(1-z)$ and set $z = e^{-2ix}$):

$$\int_0^{\frac{\pi}{2}} x \ln\left(1 - e^{-2ix}\right) dx = -\int_0^{\frac{\pi}{2}} x \sum_{k=1}^{\infty} \frac{e^{-i2kx}}{k} dx = -\sum_{k=1}^{\infty} \left(\frac{1}{k} \int_0^{\frac{\pi}{2}} x e^{-i2kx} dx \right)$$

Integration by parts easily yields the following fact:

$$\int_0^{\frac{\pi}{2}} x e^{-i2kx} dx = i\frac{\pi e^{-ik\pi}}{4k} + \frac{e^{-ik\pi} - 1}{4k^2} = i\frac{(-1)^k \pi}{4k} + \frac{(-1)^k - 1}{4k^2}$$

So the previous equation becomes:

$$\int_0^{\frac{\pi}{2}} x \ln\left(1 - e^{-2ix}\right) dx = -\sum_{k=1}^{\infty} \left(\frac{1}{k} \left(i\frac{(-1)^k \pi}{4k} + \frac{(-1)^k - 1}{4k^2} \right) \right) =$$

$$-i\frac{\pi}{4} \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} - \frac{1}{4} \sum_{k=1}^{\infty} \frac{(-1)^k}{k^3} + \frac{1}{4} \sum_{k=1}^{\infty} \frac{1}{k^3}$$

Using the easily proven facts that $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^3} = -\frac{3}{4}\zeta(3)$ and $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} = -\frac{1}{2}\zeta(2)$ (actually these are just special cases of the formula $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^n} = -\frac{2^n - 2}{2^n}\zeta(n)$ that I presented earlier in this letter) we get that the above expression is equal to:

$$-i\frac{\pi}{4}\left(-\frac{1}{2}\zeta(2)\right) - \frac{1}{4}\left(-\frac{3}{4}\zeta(3)\right) + \frac{1}{4}\zeta(3) = i\frac{\pi}{8}\zeta(2) + \frac{7}{16}\zeta(3) = i\frac{\pi^3}{48} + \frac{7}{16}\zeta(3)$$

So we have that:

$$\int_0^{\frac{\pi}{2}} x \ln\left(1 - e^{-2ix}\right) dx = \frac{7}{16}\zeta(3) + i\frac{\pi^3}{48}$$

Plugging this into Equation 2 gives us that:

$$\int_0^{\frac{\pi}{2}} x\ln\left(e^{ix}-e^{-ix}\right)dx = i\frac{\pi^3}{24}+\int_0^{\frac{\pi}{2}} x\ln\left(1-e^{-2ix}\right)dx = i\frac{\pi^3}{24}+\left(\frac{7}{16}\zeta(3)+i\frac{\pi^3}{48}\right)=$$

$$\int_0^{\frac{\pi}{2}} x\ln\left(e^{ix}-e^{-ix}\right)dx = \frac{7}{16}\zeta(3)+i\frac{\pi^3}{16}$$

Plugging this into Equation 1 gives us that:

$$\int_0^{\frac{\pi}{2}} x\ln(\sin(x))\,dx = \int_0^{\frac{\pi}{2}} x\ln\left(e^{ix}-e^{-ix}\right)dx - \left(\ln(2)+\frac{\pi}{2}i\right)\frac{\pi^2}{8}=$$

$$\left(\frac{7}{16}\zeta(3)+i\frac{\pi^3}{16}\right)-\left(\ln(2)+\frac{\pi}{2}i\right)\frac{\pi^2}{8}=\frac{7}{16}\zeta(3)-\frac{\pi^2}{8}\ln(2)+i\frac{\pi^3}{16}-i\frac{\pi^3}{16}=$$

$$\int_0^{\frac{\pi}{2}} x\ln(\sin(x))\,dx = \frac{7}{16}\zeta(3)-\frac{\pi^2}{8}\ln(2)$$

And so we get the result that we wanted:

$$\zeta(3)=\frac{2\pi^2}{7}\ln(2)+\frac{16}{7}\int_0^{\frac{\pi}{2}} x\ln(\sin(x))\,dx$$

As mentioned before this is equivalent to the statement:

$$\frac{1}{1^3}+\frac{1}{3^3}+\frac{1}{5^3}+\cdots=\frac{\pi^2}{4}\ln(2)+2\int_0^{\frac{\pi}{2}} x\ln(\sin(x))\,dx$$

I find all of this really cool. My professor once mentioned that all of this can be rigorously justified using complex analysis (notice I did interchange a sum and an integral at one point), and I kind of do see how. In any case, we have just re-derived a result due to Euler. He would have been proud.

That Was a Bit Weird

I was a little bit confused about why on page 148 you say that "we will derive it [you were referring to $\zeta(2)=\frac{\pi^2}{6}$], too, later in this section," when you had already derived the result:

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{8}$$

On page 33 of your book. It is only a couple of a quick steps away (which you actually do on page 153) to show that:

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{3}{4}\zeta(2)$$

Or more generally:

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^n} = \frac{2^n - 1}{2^n}\zeta(2)$$

So after page 33 you should immediately have had the result $\zeta(2) = \frac{\pi^2}{6}$. In any case, both of the derivations on pages 32 – 33 and pages 150 – 153 are great.

Fermat's Last Theorem

I wanted to say a quick comment on Fermat's Last Theorem to which you devote a couple of pages in your book to. Not too long ago I finally solved Fermat's Last Theorem $n = 4$ case. I never read a proof of it before and I finally, after a long time of trying, solved it. It was a great personally triumph for me. I actually solved a little bit stronger Diophantine equation. I proved the following:

"There do not exist integers $a, b, c > 0$ and $k \geq 0$ such that: $a^4 + 2^k b^4 = c^4$"

Fermat's Last Theorem $n = 4$ case is the special case of $k = 0$. Later I used the same techniques as I used to prove the above fact in order to derive equations that generate all Pythagorean triples. Pythagorean triples are triples of non-zero integers a, b, c such that $a^2 + b^2 = c^2$ (like for example $3^2 + 4^2 = 5^2$). I'm still working on other special cases of Fermat's Last Theorem. Then maybe I'll move onto the big theorem ☺!

Fourier Series

We are coming to the last few sections and indeed they are a climax of all of the mathematics done so far in this letter. The foremost thing that I liked about Fourier series when I first read about them was that they allow people to calculate the values of the Riemann-Zeta function at even integer. All you have to do is start with the right Fourier series, integrate over and over while plugging in the correct constant of integrations in order to get more and more values of the Riemann-Zeta function at even integers. But Fourier series can do so much more. Let me show you what else I discovered that they can do. We will start with the bravest and the boldest of calculations.

I absolutely loved the calculations that you presented on pages 134 – 135 on how Euler was able to discover his very own Fourier series through geometric sums. In fact I believe that it was brilliant to use this calculation as an introduction to the topic of Fourier series. Let me show you how one can use this same exact calculation in order to derive a pretty cool integral.

I discovered the following calculation while eating lunch in the Math Graduate Lounge (I pretend to be a graduate student there) at UW. Let us start with a variant of the problem that Euler considered on page 134 of your book. Specifically let us consider and calculate the sum:

$$S(t) = e^{it} + e^{3it} + e^{5it} + \cdots = \sum_{k=0}^{\infty} e^{(2k+1)it}$$

Let us use the standard trick for geometric sums in order to take the sum of the above series. We have that:

$$e^{2it}S(t) = e^{3it} + e^{5it} + e^{7it} + \cdots = \sum_{k=1}^{\infty} e^{(2k+1)it}$$

So:

$$S(t) - e^{2it}S(t) = \left(1 - e^{2it}\right)S(t) = e^{it}$$

And so we get that:

$$S(t) = \frac{e^{it}}{(1 - e^{2it})}$$

Let us turn this into a more convenient form. Multiplying the top and bottom of the fraction on the right by $\left(1 - e^{-2it}\right)$ gives us:

$$S(t) = \frac{e^{it}}{(1 - e^{2it})} = \frac{1}{\frac{1 - e^{2it}}{e^{it}}} = \frac{1}{e^{-it} - e^{it}} = \frac{1}{2i} \cdot \frac{1}{\frac{e^{-it} - e^{it}}{2i}} = \frac{-i}{2} \cdot \frac{1}{-\sin(t)} =$$

$$\frac{1}{2} i \frac{1}{\sin(t)} = \frac{1}{2} i \csc(t)$$

And so we have that:

$$S(t) = \sum_{k=0}^{\infty} e^{(2k+1)it} = \frac{1}{2} i \csc(t)$$

Let us integrate both sides of the last equality and get that (here I use the fact from high school calculus that $\int \csc(x)\, dx = -\ln|\csc(x) + \cot(x)| + C$):

$$\frac{1}{i} \sum_{k=0}^{\infty} \frac{e^{(2k+1)it}}{2k+1} = -\frac{1}{2} i \ln|\csc(t) + \cot(t)| + C$$

In order to determine the constant of integration C, just plug in $t = \pi/2$ into both sides to get that:

$$\frac{1}{i}\sum_{k=0}^{\infty}\frac{(-1)^k i}{2k+1} = \frac{\pi}{4} = 0 + C$$

(Here I used the fact that $\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1} = \frac{\pi}{4}$). So we get that $C = \frac{\pi}{4}$. Plugging this into our previous result finally gives us the Fourier series:

$$-\frac{1}{2}i\ln|\csc(t) + \cot(t)| + \frac{\pi}{4} = \frac{1}{i}\sum_{k=0}^{\infty}\frac{e^{(2k+1)it}}{2k+1}$$

Let us multiply through by i in order to make the above result look just a bit nicer:

$$\frac{1}{2}\ln|\csc(t) + \cot(t)| + \frac{\pi}{4}i = \sum_{k=0}^{\infty}\frac{e^{(2k+1)it}}{2k+1}$$

Now that we have this result, let us do something with it. Take the conjugate of both sides to get that:

$$\frac{1}{2}\ln|\csc(t) + \cot(t)| - \frac{\pi}{4}i = \sum_{k=0}^{\infty}\frac{e^{-(2k+1)it}}{2k+1}$$

Now let us multiply the last two equations by each other and integrate from 0 to 2π. Multiplying the left sides of the last two equations and then integrating from 0 to 2π gives us:

$$\int_0^{2\pi}\left(\frac{1}{2}\ln|\csc(t) + \cot(t)| + \frac{\pi}{4}i\right)\left(\frac{1}{2}\ln|\csc(t) + \cot(t)| - \frac{\pi}{4}i\right)dx =$$

$$\int_0^{2\pi}\left(\frac{1}{4}[\ln|\csc(t) + \cot(t)|]^2 + \frac{\pi^2}{16}\right)dx = \frac{1}{4}\int_0^{2\pi}[\ln|\csc(t) + \cot(t)|]^2 dx + \frac{\pi^3}{8}$$

On the other hand multiplying the right hand sides of the two equations and then integrating from 0 to 2π gives us:

$$\int_0^{2\pi}\left(\sum_{k=0}^{\infty}\frac{e^{(2k+1)it}}{2k+1}\right)\left(\sum_{n=0}^{\infty}\frac{e^{-(2n+1)it}}{2n+1}\right)dx = \int_0^{2\pi}\sum_{k=0,n=0}^{k=\infty,n=\infty}\left(\frac{e^{(2k+1)it}}{2k+1}\cdot\frac{e^{-(2n+1)it}}{2n+1}\right)dx =$$

$$\sum_{k=0,n=0}^{k=\infty,n=\infty}\left(\int_0^{2\pi}\frac{e^{(2k+1)it}}{2k+1}\cdot\frac{e^{-(2n+1)it}}{2n+1}dx\right) = \sum_{k=0}^{\infty}\frac{2\pi}{(2k+1)^2} = 2\pi\frac{3}{4}\zeta(2) = 2\pi\frac{3}{4}\cdot\frac{\pi^2}{6} = \frac{\pi^3}{4}$$

So equating the two sides gives us that:

$$\frac{1}{4}\int_{0}^{2\pi} [\ln|\csc(t) + \cot(t)|]^2 dx + \frac{\pi^3}{8} = \frac{\pi^3}{4}$$

Rearranging finally gives us the beautiful result:

$$\int_{0}^{2\pi} [\ln|\csc(t) + \cot(t)|]^2 dx = \frac{\pi^3}{2}$$

We can simplify this result even further. Noticing that by the periodicity of the trigonometric functions $\csc(t)$ and $\cot(t)$, the area under the curve of the function $[\ln|\csc(t) + \cot(t)|]^2$ is the same on the intervals $[0, \pi/2]$, $[\pi/2, \pi]$, $[\pi, 3\pi/2]$, $[3\pi/2, 2\pi]$. Since there are four intervals of these kinds we get that our beautiful result can be rewritten as:

$$\int_{0}^{\frac{\pi}{2}} [\ln|\csc(t) + \cot(t)|]^2 dx = \frac{\pi^3}{8}$$

I think that this is an absolutely fascinating and non-trivial integral equation. I remember that I showed this equation once to a group of graduate math students during lunch. The next day I asked whether anyone was able to prove the above statement and one student told me that he was able to prove it "by integration by parts, and then lots and lots of u-substitutions, lots and lots of u-substitutions." I came to the conclusion that he was probably lying. To prove the above result without having known that it came from Fourier series would require some intense complex analysis at least.

Anyways, by similar methods one can also derive the pretty fact that:

$$K = \frac{1}{2}\int_{0}^{\frac{\pi}{2}} \ln|\csc(t) + \cot(t)|\, dt$$

Where K again is Catalan's constant.

The Connection between Complex Analysis and Fourier Series

I have done a lot of complex analysis and Fourier series computations before and whenever I would do them I would always feel like there is an uncanny connection between the two. It took me yet another lunch break to help me find out why I kept on getting these uncanny chills. As it turned out, Dr. Euler's fabulous formula turned out to be the cure.

One day while eating my lunch (I took some delicious food that my mom cooked) I wrote out the Taylor series of a function $f(z)$ centered at z_0:

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k$$

where z and z_0 are of course complex numbers. Let us shift the function $f(z)$ in the direction of $-z_0$. In other words, make the change of variables $(z - z_0) \to \omega$ in order to get:

$$f(z_0 + \omega) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!} \omega^k$$

Where ω is again a complex variable. Essentially we shifted the Taylor series to the origin. Now, let us plug in $re^{i\theta}$, where $r > 0$ is a small real number, into ω in the above equation. We will then get that:

$$f\left(z_0 + re^{i\theta}\right) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!} r^k e^{ik\theta}$$

This is a Fourier series! Now let us multiply both sides by $r^{-n}e^{-in\theta}$ and integrate from 0 to 2π. We will get:

$$\int_0^{2\pi} f\left(z_0 + re^{i\theta}\right) r^{-n} e^{-in\theta} d\theta = \int_0^{2\pi} \left(\sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!} r^k e^{ik\theta} \right) r^{-n} e^{-in\theta} d\theta =$$

$$\int_0^{2\pi} \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!} r^{k-n} e^{i(k-n)\theta} d\theta = \sum_{k=0}^{\infty} \left(\frac{f^{(k)}(z_0)}{k!} r^{k-n} \int_0^{2\pi} e^{i(k-n)\theta} d\theta \right) = 2\pi \frac{f^{(n)}(z_0)}{n!}$$

And so we have:

$$\int_0^{2\pi} f\left(z_0 + re^{i\theta}\right) r^{-n} e^{-in\theta} d\theta = 2\pi \frac{f^{(n)}(z_0)}{n!}$$

Now we do the really important step. If we set $z = z_0 + re^{i\theta}$, we can then rewrite the left integral as a contour integral over a circle in the complex plane of radius r centered at z_0. With this substitution we get that $dz = ire^{i\theta} d\theta$ and so the left integral becomes ($\partial B_r(z_0)$ means the circle of radius r centered at z_0):

$$\int_0^{2\pi} \frac{f\left(z_0 + re^{i\theta}\right)}{r^n e^{in\theta}} d\theta = \frac{1}{i} \int_0^{2\pi} \frac{f\left(z_0 + re^{i\theta}\right)}{r^{n+1} e^{i(n+1)\theta}} ire^{i\theta} d\theta = \frac{1}{i} \oint_{\partial B_r(z_0)} \frac{f(z)}{(z - z_0)^{n+1}} dz = 2\pi \frac{f^{(n)}(z_0)}{n!}$$

And rearranging finally gives us the amazing result:

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_{\partial B_r(z_0)} \frac{f(z)}{(z - z_0)^{n+1}} dz$$

Called Cauchy's Formula. Of course we don't need to stick with integrating only over $\partial B_r(z_0)$. We can integrate over any contour γ such that $f(z)$ is analytic in the region enclosed by γ (and on γ itself) and such that this region includes z_0. So we can rewrite Cauchy's Formula in the more general form:

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_{\gamma} \frac{f(z)}{(z - z_0)^{n+1}} dz$$

How exciting! All of this shows how integration in the complex plane is closely related to Taylor series expansions and it works on concepts that are very much related to Fourier series. For example, one of the biggest analogies between complex analysis and Fourier series that makes all of the above possible are the facts that:

For integers m and n, $\int_0^{2\pi} e^{i(m-n)x} dx = \begin{cases} 0 & if \quad m \neq n \\ 2\pi & if \quad m = n \end{cases}$

and

For integers m and n, $\oint_{\partial u} \frac{1}{z^{m-n}} dz = \begin{cases} 0 & if \quad m \neq n-1 \\ 2\pi i & if \quad m = n-1 \end{cases}$

One Last Integral

The Fourier transform has a beautiful application to the evaluation of certain integrals through Rayleigh's energy formula. One of the amazing examples of this that you present in your book is the formula:

$$\int_0^{\infty} \frac{\sin^2(x)}{x^2} dx = \frac{\pi}{2}$$

The other integrals that you derive in your book using the energy formula can also be computed using complex analysis. The above integral equation however doesn't seem to lend itself to complex integration as easily. I however did finally figure out how to use complex analysis to compute the above integral, and so let me show you how I did it.

Again, I believe that I discovered the following calculation while eating lunch.

I started off by pulling off a variant of a famous trick: let us construct the function $I(\alpha)$:

$$I(\alpha) = \int_0^{\infty} e^{-\alpha x^2} \frac{\sin^2(x)}{x^2} dx$$

To calculate the integral $\int_0^{\infty} \frac{\sin^2(x)}{x^2} dx$, all we have to do is simply evaluate $I(\alpha)$ at $\alpha = 0$. Simple, right? But how do we do that? We first differentiate our function $I(\alpha)$ (In the second step here I will pull the derivative in under the integral sign):

$$\frac{dI(\alpha)}{d\alpha} = \frac{d}{d\alpha}\left(\int_0^\infty e^{-\alpha x^2}\frac{\sin^2(x)}{x^2}dx\right) = \int_0^\infty \frac{\partial}{\partial\alpha}\left(e^{-\alpha x^2}\frac{\sin^2(x)}{x^2}\right)dx =$$

$$-\int_0^\infty e^{-\alpha x^2}(\sin(x))^2 dx = -\int_0^\infty e^{-\alpha x^2}\left(\frac{e^{ix}-e^{-ix}}{2}\right)^2 dx = \frac{1}{4}\int_0^\infty e^{-\alpha x^2}\left(e^{2ix}+e^{-2ix}-2\right)dx =$$

Equation 3: $$\frac{dI(\alpha)}{d\alpha} = \frac{1}{4}\int_0^\infty e^{-\alpha x^2+2ix}dx + \frac{1}{4}\int_0^\infty e^{-\alpha x^2-2ix}dx - \frac{1}{2}\int_0^\infty e^{-\alpha x^2}dx$$

Now, let us analyze the sum of the first two integrals in this last equation. Let us complete the square in the exponents in each of these integrands:

$$\frac{1}{4}\int_0^\infty e^{-\alpha x^2+2ix}dx + \frac{1}{4}\int_0^\infty e^{-\alpha x^2-2ix}dx =$$

$$\frac{1}{4}e^{-\frac{1}{\alpha}}\int_0^\infty e^{-\alpha x^2+2ix+\frac{1}{\alpha}}dx + \frac{1}{4}e^{-\frac{1}{\alpha}}\int_0^\infty e^{-\alpha x^2-2ix+\frac{1}{\alpha}}dx =$$

$$\frac{1}{4e^{\frac{1}{\alpha}}}\left[\int_0^\infty e^{-\left(\sqrt{\alpha}x-\frac{i}{\sqrt{\alpha}}\right)^2}dx + \int_0^\infty e^{-\left(\sqrt{\alpha}x+\frac{i}{\sqrt{\alpha}}\right)^2}dx\right]$$

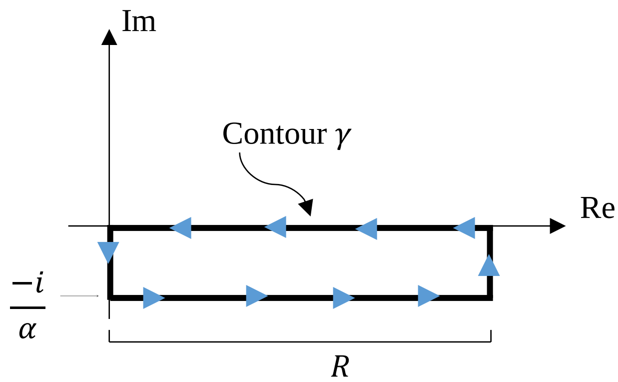
And so we get that:

Equation 4: $$\frac{1}{4}\int_0^\infty e^{-\alpha x^2+2ix}dx + \frac{1}{4}\int_0^\infty e^{-\alpha x^2-2ix}dx = \frac{1}{4e^{\frac{1}{\alpha}}}\left[\int_0^\infty e^{-\left(\sqrt{\alpha}x-\frac{i}{\sqrt{\alpha}}\right)^2}dx + \int_0^\infty e^{-\left(\sqrt{\alpha}x+\frac{i}{\sqrt{\alpha}}\right)^2}dx\right]$$

Let us first calculate the first integral on the right hand side:

$$\int_0^\infty e^{-\left(\sqrt{\alpha}x-\frac{i}{\sqrt{\alpha}}\right)^2}dx$$

How should we approach calculating the above integral? We look to the complex plane for help. Let us take the contour integral of the function $e^{-(\sqrt{\alpha}z)^2} = e^{-\alpha z^2}$ on the following contour in the complex plane:



Let us call the above contour γ and its width R. Suppose that we are integrating the function $e^{-\alpha z^2}$ counterclockwise on the above contour. By breaking up the contour integral into the four integrals on the sides of the rectangle above we get that:

$$\oint_{\gamma} e^{-\alpha z^2} dz = i\int_0^{-\frac{1}{\alpha}} e^{\alpha y^2} dy + \int_0^R e^{-\alpha\left(x-\frac{i}{\alpha}\right)^2} dx + i\int_{-\frac{1}{\alpha}}^0 e^{-\alpha(R+iy)^2} dy + \int_R^0 e^{-\alpha x^2} dx$$

Now, since e^{-z^2} is an analytic function and γ is a closed loop, we get that the contour integral $\oint_{\gamma} e^{-z^2} dz$ is equal to zero. So the left hand side of the last equation is zero and so we have:

$$i\int_0^{-\frac{1}{\alpha}} e^{\alpha y^2} dy + \int_0^R e^{-\left(\sqrt{\alpha}x-\frac{i}{\sqrt{\alpha}}\right)^2} dx + i\int_{-\frac{1}{\alpha}}^0 e^{-\alpha(R+iy)^2} dy - \int_0^R e^{-\alpha x^2} dx = 0$$

(In the exponent of the second integrand I pulled the α into the parentheses. I also flipped the limits on the last integral, which introduces a minus sign) Now let us stretch the rectangular contour in the right direction to infinity. In other words, let $R \to \infty$. The third integral in the last expression goes to zero as $R \to \infty$ and so as we take the limit of both sides as $R \to \infty$ we get that:

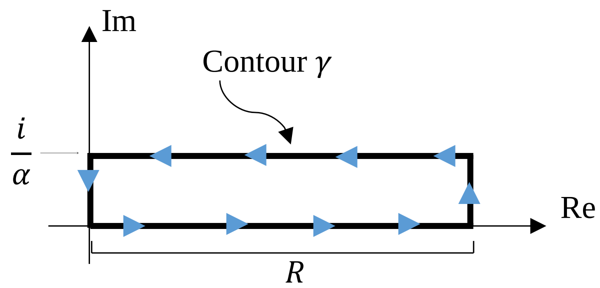
$$i\int_0^{-\frac{1}{\alpha}} e^{\alpha y^2} dy + \int_0^{\infty} e^{-\left(\sqrt{\alpha}x-\frac{i}{\sqrt{\alpha}}\right)^2} dx - \int_0^{\infty} e^{-\alpha x^2} dx = 0$$

And so we get that:

Equation 5:

$$\int_0^{\infty} e^{-\left(\sqrt{\alpha}x-\frac{i}{\sqrt{\alpha}}\right)^2} dx = -i\int_0^{-\frac{1}{\alpha}} e^{\alpha y^2} dy + \int_0^{\infty} e^{-\alpha x^2} dx$$

If you do the same type of calculation of integrating $e^{-\alpha z^2}$ but on the contour:



You will get a similar result involving the other integral:

$$\int_0^{\infty} e^{-\left(\sqrt{\alpha}x+\frac{i}{\sqrt{\alpha}}\right)^2} dx = -i\int_0^{\frac{1}{\alpha}} e^{\alpha y^2} dy + \int_0^{\infty} e^{-\alpha x^2} dx$$

Since $\int_0^{\frac{1}{\alpha}} e^{\alpha y^2} dy = -\int_0^{-\frac{1}{\alpha}} e^{\alpha y^2} dy$ by symmetry of $e^{\alpha z^2}$, we get that the above expression is equivalent to:

Equation 6:
$$\int_0^{\infty} e^{-\left(\sqrt{\alpha}x+\frac{i}{\sqrt{\alpha}}\right)^2} dx = i\int_0^{-\frac{1}{\alpha}} e^{\alpha y^2} dy + \int_0^{\infty} e^{-\alpha x^2} dx$$

Now let us plug in the results Equation 5 and Equation 6 into Equation 4. We will then get that:

$$\frac{1}{4}\int_0^{\infty} e^{-\alpha x^2+2ix} dx + \frac{1}{4}\int_0^{\infty} e^{-\alpha x^2-2ix} dx =$$

$$\frac{1}{4e^{\frac{1}{\alpha}}}\left[-i\int_0^{-\frac{1}{\alpha}} e^{\alpha y^2} dy + \int_0^{\infty} e^{-\alpha x^2} dx + i\int_0^{-\frac{1}{\alpha}} e^{\alpha y^2} dy + \int_0^{\infty} e^{-\alpha x^2} dx\right] = \frac{1}{4e^{\frac{1}{\alpha}}}\left[2\int_0^{\infty} e^{-\alpha x^2} dx\right] =$$

$$\frac{1}{2e^{1/\alpha}}\int_0^{\infty} e^{-\alpha x^2} dx$$

And so we get the equation:

$$\frac{1}{4}\int_0^{\infty} e^{-\alpha x^2+2ix} dx + \frac{1}{4}\int_0^{\infty} e^{-\alpha x^2-2ix} dx = \frac{1}{2e^{1/\alpha}}\int_0^{\infty} e^{-\alpha x^2} dx$$

Plugging this result into Equation 3 gives us:

$$\frac{dI(\alpha)}{d\alpha} = \frac{1}{2e^{1/\alpha}}\int_0^\infty e^{-\alpha x^2}dx - \frac{1}{2}\int_0^\infty e^{-\alpha x^2}dx = \frac{1}{2}\left(\frac{1}{e^{\frac{1}{\alpha}}} - 1\right)\int_0^\infty e^{-\alpha x^2}dx$$

Now we know that the integral $\int_0^\infty e^{-\alpha x^2}dx = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}$. This then give us that:

$$\frac{dI(\alpha)}{d\alpha} = \frac{1}{2}\left(\frac{1}{e^{\frac{1}{\alpha}}} - 1\right)\frac{1}{2}\sqrt{\frac{\pi}{\alpha}}$$

And so:

$$\frac{dI(\alpha)}{d\alpha} = \frac{\sqrt{\pi}}{4\sqrt{\alpha}}\left(e^{-\frac{1}{\alpha}} - 1\right)$$

Now, what do we do from here? How do we use this expression to get the value of $\int_0^\infty \frac{\sin^2(x)}{x^2}dx$. Simple! We integrate both sides of the last equation from 0 to ∞:

$$\int_0^\infty \frac{dI(\alpha)}{d\alpha}d\alpha = I(\infty) - I(0) = \frac{\sqrt{\pi}}{4}\int_0^\infty \left(e^{-\frac{1}{\alpha}} - 1\right)\frac{1}{\sqrt{\alpha}}d\alpha$$

(The $d\alpha$'s cancel out in the leftmost integral ☺) One can easily notice (and prove) that:

$$I(\infty) = \lim_{\alpha\to\infty}\left(\int_0^\infty e^{-\alpha x^2}\frac{\sin^2(x)}{x^2}dx\right) = 0$$

And so we have that:

$$-I(0) = -\int_0^\infty \frac{\sin^2(x)}{x^2}dx = \frac{\sqrt{\pi}}{4}\int_0^\infty \left(e^{-\frac{1}{\alpha}} - 1\right)\frac{1}{\sqrt{\alpha}}d\alpha$$

Or more elegantly written:

$$\int_0^\infty \frac{\sin^2(x)}{x^2}dx = \frac{\sqrt{\pi}}{4}\int_0^\infty \left(1 - e^{-\frac{1}{\alpha}}\right)\frac{1}{\sqrt{\alpha}}d\alpha$$

But this is just one integral written as another integral, how does this help us (what did you expect, that's what complex integration does essentially: write one integral as another)? But notice that $\left(1 - e^{-\frac{1}{\alpha}}\right)\frac{1}{\sqrt{\alpha}}$ is a pretty interesting looking integrand. Let us do some change of variables to turn this into something more familiar. First set $u = \sqrt{\alpha}$ and so the integral on the right will become:

$$\frac{\sqrt{\pi}}{2}\int_0^\infty \left(1-e^{-\frac{1}{u^2}}\right)du$$

This is an even more interesting looking integrand! Remind you of anything? It should! Let us now do the substitution $y=\frac{1}{u}$ so that the above integral becomes:

$$\frac{\sqrt{\pi}}{2}\int_0^\infty \frac{\left(1-e^{-y^2}\right)}{y^2}dy$$

Now do integration by parts:

$$\frac{\sqrt{\pi}}{2}\int_0^\infty \frac{\left(1-e^{-y^2}\right)}{y^2}dy=\frac{\sqrt{\pi}}{2}\left[\left(-\frac{1-e^{-y^2}}{y}\right)\Bigg|_0^\infty+2\int_0^\infty e^{-y^2}dy\right]=$$

$$\sqrt{\pi}\int_0^\infty e^{-y^2}dx=\sqrt{\pi}\left(\frac{1}{2}\sqrt{\pi}\right)=\frac{\pi}{2}$$

So we finally get that (drum rolls please):

$$\int_0^\infty \frac{\sin^2(x)}{x^2}dx=\frac{\pi}{2}$$

Amazing! Right?

Although I have shown you a derivation of the above integral using complex analysis, there is a much, much more simple derivation of the above integral equation. Simply integrate by parts to get that (in the last step I do the substitution $u=2x$):

$$\int_0^\infty \frac{\sin^2(x)}{x^2}dx=\left(-\frac{\sin^2(x)}{x}\right)\Bigg|_0^\infty+\int_0^\infty \frac{2\sin(x)\cos(x)}{x}dx=\int_0^\infty \frac{\sin(2x)}{x}dx=\int_0^\infty \frac{\sin(u)}{u}du$$

And on pages 64 – 65 you give a very simple proof of the fact that: $\int_0^\infty \frac{\sin(x)}{x}dx=\frac{\pi}{2}$. Thus we do have a simple proof of the equation:

$$\int_0^\infty \frac{\sin^2(x)}{x^2}dx=\frac{\pi}{2}$$

I point this out because it proves that you were wrong on page 209 of your book when you say that this equation "is *not* easily derived by other means."

I showed you the complex analysis derivation before the integration by parts proof because had I showed you the second much simpler proof first, you probably would not have paid attention to the derivation involving complex analysis. ☺

Concluding Words

I really liked your two books "An Imaginary Tale: The Story of $\sqrt{-1}$" and "Dr. Euler's Fabulous Formula." I have learned so much from these books and they have given me starting points from which I can go and invent new mathematics. From these two books I have learned about topics that my math courses at the University of Washington wouldn't ever dream of covering. Part of the reason for this is that they have to prove everything rigorously (which is right because math is rigor) and are never brave enough to cover material that require Eulerian boldness in their derivations. Some of the topics are inaccessible because they require much more advanced mathematics in order to prove 100% rigorously. But with these books and a little bit of reliance on intuition, I have been able to learn how to arrive at so many surprising, and beautiful results.

Now that it's summer, it's time to enjoy the outdoors! Have a great summer!

Thank you.

Sincerely,

Haim Grebnev